MATH 200 – MIDTERM 2 – PRACTICE QUESTIONS

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Note: Here are some practice questions for the exam, and I'm hoping that they are representative of the difficulty of the exam. Just like last time, the midterm will have 4 questions, but this time I'm expecting the questions to all have the same length. I would like to remind you that the actual exam covers not just those practice questions, but also everything covered in lecture, in the book, and the homework. Please refer to the study guide if you want to know exactly what to study.

Date: Friday, April 21, 2017.

1. (Again, note that there might not be a T/F portion on the exam)

   Label the following statements as **TRUE (T)** or **FALSE (F)**. Any correct answer gives you 4 points, and any incorrect answer gives you 0 points. You do **NOT** get points off for an incorrect answer, and you do **NOT** need to justify your answers.

   (a) The union of *any* number of countable sets is countable.

   (b) The intersection of *any* number of countable sets is countable

   (c) For any f, and any set C we have that $f^{-1}(f(C)) = C$, even if f isn't invertible.

   (d) If $p \sim q$ means $p \Rightarrow q$, then $\sim$ is an equivalence relation on the set of all logical statements.

   (e) $A = \{x \mid x \notin A\}$ is a set.

2. Show by strong induction that every positive integer n can be written as a sum of distinct powers of two.

   For example, we can write $11 = 2^0 + 2^1 + 2^3$.

   Hint: Argue in terms of whether n is even or not.

3. A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:

   (1) Rabbit pairs are not fertile during their first month of life but thereafter give birth to four new male/female pairs at the end of the month.
   (2) No rabbits die

   Let r_n = number of pairs of rabbits alive at the end of month n, with $r_0 = 1$. Find a recurrence relation for r_n, and briefly explain how you got your answer.

4. Suppose you're in the set-up of the Tower of Hanoi problem with three poles A, B, C, but this time you have a super-tower of size n, which consists of n disks of size n topped with $n-1$ disks of size $n-1, \cdots$, topped with 2 disks of size 2 and finally 1 disk of size 1. Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, it's ok if a disk is placed on top of a disk of the same size. Let c_n be the minimum number of moves needed to transfer a super-tower of size n from one pole to another. Find a recurrence relation for c_n and briefly explain how you got your answer.

5. Find the general solution to the following difference equations:

(a) $u_{n+2} = 6u_{n+1} - 9u_n$
(b) $u_{n+2} = 7u_{n+1} - 10u_n$

6. Solve the following difference equation

$$\begin{cases} F_{n+2} = F_{n+1} + F_n \\ F_0 = 0, F_1 = 1 \end{cases}$$

7. Prove or disprove the following statements:

   (a) $(A - B) - C = A - (B - C)$

   (b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

   (c) $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

8. Let $A = \{1, 2, 3, 4, 5\}$, and define a function $F : \mathcal{P}(A) \to \{0, 1\}$ as follows:

$$F(X) = \begin{cases} 0 & \text{if } X \text{ has an even number of elements} \\ 1 & \text{if } X \text{ has an odd number of elements} \end{cases}$$

(a) Is F one-to-one?

(b) Is F onto $\{0, 1\}$?

9. Prove that $\mathbb{R}$ is uncountable

10. Define $F : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ by the rule

$$F(n) = \begin{cases} 1 & \text{if } n = 1 \\ F\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ 1 + F(5n - 9) & \text{if } n \text{ is odd} \end{cases}$$

Is F a function?

11. Show that if $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.

12. Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ that is one-to-one but not onto; an example that is onto but not one-to-one; and an example that is one-to-one and onto but not $f(n) = n$ or $f(n) = -n$. You don't have to come up with formulas, graphs are ok too.

13. Define the relation $\sim$ on $A = \mathbb{R}$ by $x \sim y$ if and only if $x - y$ is rational. Show that $\sim$ is an equivalence relation.