SOLUTIONS

MATH 200 – MIDTERM 2 – PRACTICE QUESTIONS

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Note: Here are some practice questions for the exam, and I'm hoping that they are representative of the difficulty of the exam. Just like last time, the midterm will have 4 questions, but this time I'm expecting the questions to all have the same length. I would like to remind you that the actual exam covers not just those practice questions, but also everything covered in lecture, in the book, and the homework. Please refer to the study guide if you want to know exactly what to study.

Date: Friday, April 21, 2017.

1. **Label the following statements as TRUE (T) or FALSE (F). Any correct answer gives you 2 points, and any incorrect answer gives you 0 points. You do NOT get points off for an incorrect answer, and you do NOT need to justify your answers.**

Ⓕ (a) The union of any number of countable sets is countable.

Ⓣ (b) The intersection of any number of countable sets is countable

Ⓕ (c) For any f, AND FOR ANY C we have that $f^{-1}(f(\overset{C}{\cancel{A}})) = \overset{C}{\cancel{A}}$, even if f isn't invertible.

Ⓕ (d) The relation $p \sim q$ if and only if $p \Rightarrow q$ is an equivalence relation on the set of all statements.

Ⓕ (e) $A = \{x \mid x \notin A\}$ is a set.

EXPLANATIONS (OPTIONAL)

Ⓕ (a) EX NOTICE THAT $\mathbb{R}$ (UNCOUNTABLE) IS THE UNION OF 1-ELEMENT SETS OF THE FORM $\{x\}$, WHERE $x \in \mathbb{R}$, THAT IS

$$\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}$$

EACH SET $\{x\}$ IS COUNTABLE (FINITE!)

BUT $\mathbb{R}$ ISN'T COUNTABLE.

(FOR ANOTHER EX, SEE PROBLEM 11 (c))

(THE POINT IS THE UNION HAS TO BE COUNTABLE)

Ⓣ (b) THE INTERSECTION IS INCLUDED IN ONE OF THE SETS (EX $A \cap B \subseteq A$), AND THE LATTER IS COUNTABLE, HENCE THE INTERSECTION IS A SUBSET OF A COUNTABLE SET, AND THEREFORE COUNTABLE.

Ⓕ (c) TAKE $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ AND $C = [0, \infty)$ (THERE ARE MANY OTHER EXAMPLES)

THEN $f(C) = f([0,\infty)) = [0,\infty)$

AND $f^{-1}(f(C)) = f^{-1}([0,\infty)) = \{x \in \mathbb{R} \mid f(x) \in [0,\infty)\} = \mathbb{R}$

SO $f^{-1}(f(C)) = \mathbb{R} \neq [0,\infty) = C$

Ⓕ (d) IT IS NOT SYMMETRIC, $p \sim q \not\Rightarrow q \sim p$, BECAUSE $(p \Rightarrow q) \not\Rightarrow (q \Rightarrow p)$ IN GENERAL

Ⓕ (e) IF $x \in A$, THEN $x \notin A$ BY DEF OF A AND IF $x \notin A$ THEN $x \in A$ BY DEF OF A $\Rightarrow \Leftarrow$

2. Show by induction (or strong induction) that every positive integer n can be written as a sum of distinct powers of two.

For example, we can write $7 = 1 + 2 + 2^2$.

Hint: Argue in terms of whether n is even or not.

STRONG INDUCTION

LET P_N BE THE PROPOSITION

$N = 2^{a_1} + 2^{a_2} + \dots + 2^{a_M}$ FOR SOME M, WHERE ALL THE a_i ARE DISTINCT.

BASE CASE $N=1$ NOTICE THAT $1 = 2^0$, SO $M=1$ AND $a_1 = 0$ HERE

INDUCTIVE STEP SUPPOSE P_K IS TRUE FOR ALL $K = 1, \dots, N-1$, THAT IS $K = 2^{a_1} + \dots + 2^{a_M}$ FOR SOME M, WHERE ALL THE a_i ARE DISTINCT

WANT TO SHOW P_N IS TRUE, THAT IS $N = 2^{b_1} + \dots + 2^{b_\ell}$ FOR SOME ℓ, WHERE ALL THE b_i ARE DISTINC

CASE 1 (N IS EVEN), THAT IS $N = 2K$ FOR SOME $1 \leq K < N$

SINCE P_K IS TRUE, WE HAVE $K = 2^{a_1} + \dots + 2^{a_n}$ FOR SOME M, WHERE ALL THE a_i ARE DISTINCT

BUT THEN $N = 2K = 2(2^{a_1} + \dots + 2^{a_M}) = 2^{a_1+1} + \dots + 2^{(a_M)+1}$

$= 2^{b_1} + \dots + 2^{b_M}$, WHERE $b_i = a_i + 1$, WHICH ARE ALL DISTINCT BECAUSE THE a_i ARE!

SO P_N IS TRUE

CASE 2 (N IS ODD), THAT IS $N = 2K+1$ FOR SOME $1 \leq K < N$

SINCE P_K IS TRUE, WE HAVE $K = 2^{a_1} + \dots + 2^{a_M}$

SO $N = 2K+1 = 2(2^{a_1} + \dots + 2^{a_M}) + 1 = 1 + 2^{a_1+1} + \dots + 2^{a_M+1}$

$= 2^{b_1} + 2^{b_2} + \dots + 2^{b_{M+1}}$

WHERE $b_1 = 0$, $b_2 = a_1 + 1$, $b_3 = a_2 + 1$, $b_{M+1} = a_M + 1$,

WHICH ARE ALL DISTINCT BECAUSE $b_2, \dots, b_{M+1}$ ARE DISTINCT (BECAUSE THE a_i ARE)

AND $b_2, \dots, b_{M+1} \geq 1$ BUT $b_1 < 1$

SO P_N IS TRUE

IN BOTH CASES, P_N IS TRUE, HENCE P_N IS TRUE

AND THEREFORE, BY INDUCTION, P_N IS TRUE $\forall N$, THAT IS EVERY $N > 0$ CAN BE WRITTEN AS A SUM OF DISTINCT POWERS OF PRIMES.

3. A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:

   (1) Rabbit pairs are not fertile during their first month of life but thereafter give birth to four new male/female pairs at the end of the month.
   (2) No rabbits die

   Let r_n = number of pairs of rabbits alive at the end of month n, with $r_0 = 1$. Find a recurrence relation for r_n.

JUST LIKE THE FI-BUN-ACCI EXAMPLE, WE NEED TO RELATE r_{N+2} WITH r_N AND r_{N+1}

r_{N+2} = THE NUMBER OF RABBITS ALIVE AT THE END OF MONTH N+2

= (NUMBER OF RABBITS ALIVE AT THE END OF MONTH N+1) + (NUMBER OF RABBITS <u>BORN</u> AT THE END OF MONTH N+1)

= r_{N+1} + 4 (NUMBER OF <u>OLD</u> RABBITS AT THE END OF MONTH N+1)

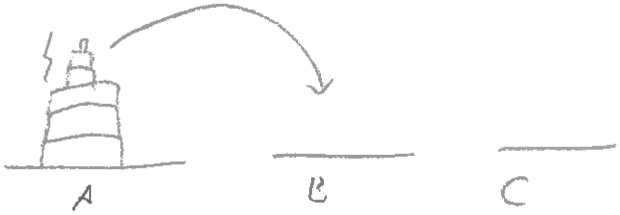
= r_{N+1} + 4 (NUMBER OF RABBITS AT THE END OF MONTH N)

= $r_{N+1} + 4 r_N$

SO $\boxed{r_{N+2} = r_{N+1} + 4 r_N}$

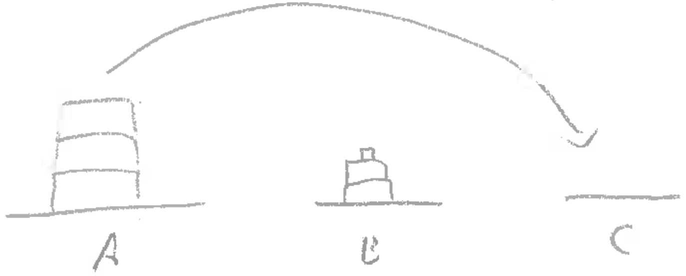
4. Suppose you're in the set-up of the Tower of Hanoi problem with three poles A, B, C, but this time you have a super-tower of size n, which consists of n disks of size n topped with $n-1$ disks of size $n-1, \cdots$, topped with 2 disks of size 2 and finally 1 disk of size 1. Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, it's ok if a disk is placed on top of a disk of the same size. Let c_n be the minimum number of moves needed to transfer a super-tower of size n from one pole to another. Find a recurrence relation for c_n and briefly explain how you got your answer.

STEP 1 TRANSFER TOP SUPER-TOWER OF SIZE N-1 FROM A TO B



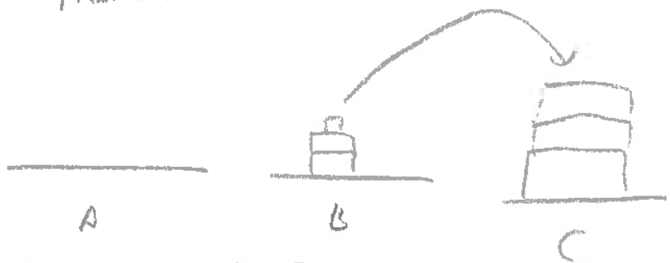
THIS TAKES C_{N-1} MOVES

STEP 2 TRANSFER BOTTOM N DISKS FROM A TO C



THIS TAKES N MOVES

STEP 3 TRANSFER SUPER-TOWER OF SIZE N-1 FROM B TO C



THIS TAKES C_{N-1} MOVES

$$C_N = \text{TOTAL \# OF MOVES} = \underset{\text{STEP 1}}{C_{N-1}} + \underset{\text{STEP 2}}{N} + \underset{\text{STEP 3}}{C_{N-1}} = 2C_{N-1} + N$$ (AND EVERY MOVE WAS NECESSARY)

$$\boxed{C_N = 2C_{N-1} + N}$$

5. Find the general solution to the following difference equations:

(a) $u_{n+2} = 6u_{n+1} - 9u_n$
(b) $u_{n+2} = 7u_{n+1} - 10u_n$

(a) $U_{N+2} - 6U_{N+1} + 9U_N = 0$

AUX $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3$ (REPEATED ROOT)

$$\Rightarrow \boxed{U_N = A3^N + BN3^N}$$

(b) $U_{N+2} - 7U_{N+1} + 10U_N = 0$

AUX $r^2 - 7r + 10 = 0$

$(r-5)(r-2) = 0$

$r = 2, 5$

$$\Rightarrow \boxed{U_N = A2^N + B5^N}$$

6 ~~3.~~ Solve the following difference equation

$$\begin{cases} F_{n+2} = F_{n+1} + F_n \\ F_0 = 0, F_1 = 1 \end{cases}$$

~~Note: I would like to remind~~ you that the cases with complex roots of the form $r = ai$ with $a \in \mathbb{R}$ (see lecture)~~, as well as~~ the case of a ~~root (Example 8.5.8) are all fair game for~~ the exam. } IGNORE

$$F_{N+2} - F_{N+1} - F_N = 0$$

AUXILIARY EQUATION $r^2 - r - 1 = 0$

$$\Rightarrow \quad r = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

SO THE GENERAL SOLUTION IS $F_N = A\left(\frac{1+\sqrt{5}}{2}\right)^N + B\left(\frac{1-\sqrt{5}}{2}\right)^N$

$$0 = F_0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0 = A + B = 0 \Rightarrow \underline{B = -A}$$

SO $F_N = A\left(\frac{1+\sqrt{5}}{2}\right)^N + (-A)\left(\frac{1-\sqrt{5}}{2}\right)^N$

$$1 = F_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + (-A)\left(\frac{1-\sqrt{5}}{2}\right) = \cancel{\frac{A}{2}} + \frac{\sqrt{5}A}{2} - \cancel{\frac{A}{2}} + \frac{\sqrt{5}A}{2} = \sqrt{5}A$$

SO $\sqrt{5}A = 1 \Rightarrow \underline{A = \frac{1}{\sqrt{5}}}$

AND THEREFORE $\boxed{F_N = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^N - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^N}$

(T) (c) (⊆) SUPPOSE $X \times Y \in \underline{\mathcal{P}(A \times B)}$, THEN $X \times Y \subseteq A \times B$

AND SO $X \subseteq A$ AND $Y \subseteq B$

AND SO $X \in \mathcal{P}(A)$ AND $Y \in \mathcal{P}(B)$

AND SO $X \times Y \in \underline{\mathcal{P}(A) \times \mathcal{P}(B)}$ ✓

(<u>WHY?</u> IF $p \in X$ AND $q \in Y$,
THEN $(p,q) \in X \times Y \subseteq A \times B$,
SO $(p,q) \in A \times B$, SO $p \in A$ AND $q \in B$

(⊇) IF $X \times Y \in \underline{\mathcal{P}(A) \times \mathcal{P}(B)}$

THEN $X \in \mathcal{P}(A)$ AND $Y \in \mathcal{P}(B)$

SO $X \subseteq A$ and $Y \subseteq B$,

SO $X \times Y \subseteq A \times B$

SO $X \times Y \in \underline{\mathcal{P}(A \times B)}$ ✓

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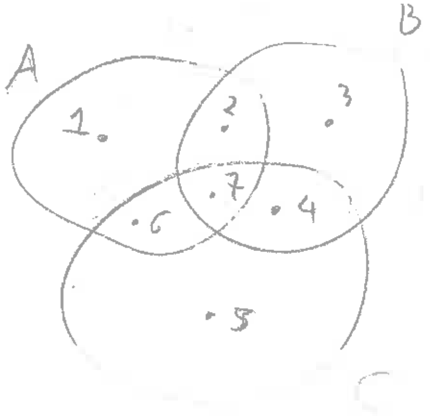
7 ~~3~~. Prove or disprove the following statements:

(a) $(A - B) - C = A - (B - C)$

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(c) $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

(F) (a)



LET $A = \{1, 2, 6, 7\}$

$B = \{2, 3, 4, 7\}$

$C = \{4, 5, 6, 7\}$

$(A - B) - C = \{1, 6\} - \{4, 5, 6, 7\} = \{1\}$

BUT $B - C = \{2, 3, 4, 7\} - \{4, 5, 6, 7\} = \{2, 3\}$

AND $A - (B - C) = \{1, 2, 6, 7\} - \{2, 3\} = \{1, 6, 7\}$

$\neq$

SO $(A - B) - C \neq A - (B - C)$

(T) (b) (⊆) SUPPOSE $(x, y) \in \underline{A \times (B \cap C)}$, THEN $x \in A$ AND $y \in B \cap C$

SO $x \in A$ AND ($y \in B$ AND $y \in C$)

SO ($x \in A$ AND $y \in B$) AND ($x \in A$ AND $y \in C$)

SO $(x, y) \in A \times B$ AND $(x, y) \in A \times C$

SO $(x, y) \in \underline{(A \times B) \cap (A \times C)}$

(⊇) SUPPOSE $\underline{(x, y) \in (A \times B) \cap (A \times C)}$

THEN $(x, y) \in (A \times B)$ AND $(x, y) \in (A \times C)$, SO ($x \in A$ AND $y \in B$) AND ($x \in A$ AND $y \in C$

SO $x \in A$ AND ($y \in B$ AND $y \in C$)

SO $x \in A$ AND $y \in B \cap C$

SO $(x, y) \in \underline{A \times (B \cap C)}$

8. Let $A = \{1,2,3,4,5\}$, and define a function $F : \mathcal{P}(A) \to \{0,1\}$ as follows:

$$F(X) = \begin{cases} 0 & \text{if } X \text{ has an even number of elements} \\ 1 & \text{if } X \text{ has an odd number of elements} \end{cases}$$

(a) Is F one-to-one?

(b) Is F onto $\{0,1\}$?

(a) F IS <u>NOT</u> ONE-TO-ONE BECAUSE EVEN THOUGH

$\{1,3\} \neq \{1,2,4,5\}$

$$F(\underbrace{\{1,3\}}_{\text{2 ELEMENTS (EVEN)}}) = F(\underbrace{\{1,2,4,5\}}_{\text{4 ELEMENTS (EVEN)}}) = 0$$

(b) <u>F IS ONTO $\{0,1\}$</u>

WE NEED TO SHOW THAT $\forall b \in \{0,1\}$ THERE IS AN $X \in \mathcal{P}(A)$ SUCH THAT $F(X) = b$

BUT $b \in \{0,1\} \Rightarrow b = 0, 1,$

SO WE NEED TO FIND 1) AN $X \in \mathcal{P}(A)$ WITH $F(X) = 0$

2) AN $X \in \mathcal{P}(A)$ WITH $F(X) = 1$

FOR 1), LET $X = \{1,2\}$, THEN $F(X) = F(\underbrace{\{1,2\}}_{\text{2 ELEMENTS (EVEN)}}) = 0$

FOR 2), LET $X = \{1\}$, THEN $F(X) = F(\underbrace{\{1\}}_{\text{1 ELEMENT (ODD)}}) = \underline{1}$

THEREFORE F IS ONTO $\{0,1\}$

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9. Prove that $\mathbb{R}$ is uncountable

1) WE WILL SHOW $(0,1)$ IS UNCOUNTABLE, WHICH IS ENOUGH, BECAUSE THEN $\mathbb{R}$ CONTAINS AN UNCOUNTABLE SUBSET AND IS THEREFORE UNCOUNTABLE.

2) SUPPOSE $(0,1)$ IS COUNTABLE, AND LIST ALL ITS ELEMENTS USING DECIMAL EXPANSIONS:

d

$$a_1 = 0.\,a_{11}\;a_{12}\;a_{13}\;\cdots$$
$$a_2 = 0.\,a_{21}\;a_{22}\;a_{23}\;\cdots$$
$$a_3 = 0.\,a_{31}\;a_{32}\;a_{33}\;\cdots$$

3) NOW LET $d = 0.\,a_{11}\,a_{22}\,a_{33}\cdots$ BE THE DIAGONAL NUMBER

AND DEFINE $X = 0.\,X_1\,X_2\,X_3\cdots$ BY $X_i \neq a_{ii}$ (ANY DIGIT EXCEPT FOR a_{ii})

4) LET N BE ARBITRARY, AND NOTICE THAT X CANNOT BE EQUAL TO a_N, BECAUSE THE N^{TH} DIGIT OF X, WHICH IS X_N IS NOT EQUAL TO THE N^{TH} DIGIT OF a_N, WHICH IS $a_{NN} \neq X_i$ (BY CONSTRUCTION!)

IT FOLLOWS THAT X IS NOT ON THE LIST (IT IS NOT EQUAL TO ANY OF THE a_N!), WHICH IS A CONTRADICTION WITH THE FACT THAT WE LISTED ALL ITS ELEMENTS.

10 ~~7~~. Define $F : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ by the rule

$$F(n) = \begin{cases} 1 & \text{if } n = 1 \\ F\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ 1 + F(5n - 9) & \text{if } n \text{ is odd} \end{cases}$$

Is F a function?

LET'S CALCULATE A FEW VALUES OF F.

$$F(1) = 1$$

$$F(2) \underset{\substack{| \\ 2 \text{ IS EVEN}}}{=} F\left(\frac{2}{2}\right) = F(1) = 1$$

$$F(3) \underset{\substack{| \\ 3 \text{ IS ODD}}}{=} 1 + F(5 \times 3 - 9) = 1 + F(6) \underset{\substack{| \\ 6 \text{ IS EVEN}}}{=} 1 + F(3)$$

AND SO $F(3) = 1 + F(3)$ AND SO $0 = 1 \Rightarrow \Leftarrow$

HENCE F CANNOT BE A FUNCTION! (IT'S NOT WELL-DEFINED)

11 **Show that if $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.**

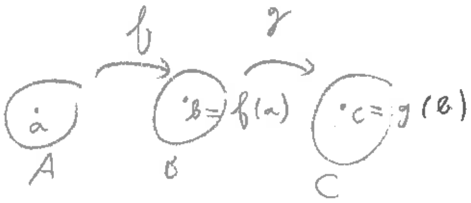
1) LET'S SHOW THAT $g \circ f$ IS ONE-TO-ONE.

SUPPOSE $(g \circ f)(x) = (g \circ f)(y)$ (SHOW $x = y$)

THEN $g(\underbrace{f(x)}) = g(\underbrace{f(y)})$

THEN $f(x) = f(y)$ SINCE g IS ONE-TO-ONE

THEN $x = y$ SINCE f IS ONE-TO-ONE



HENCE $g \circ f$ IS ONE-TO-ONE

2) LET'S SHOW THAT $g \circ f$ IS ONTO

LET $c \in C$, WANT TO SHOW $\exists a \in A$ WITH $c = (g \circ f)(a)$

BUT SINCE $c \in C$ AND g IS ONTO $\exists b \in B$ WITH $c = g(b)$

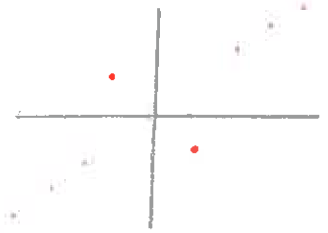
AND SINCE $b \in B$ AND f IS ONTO $\exists a \in A$ WITH $b = f(a)$

AND SO $c = g(\underline{b}) = g(f(a)) = (g \circ f)(a)$ ✓

HENCE $g \circ f$ IS ONTO

3) SINCE $g \circ f$ IS ONE-TO-ONE AND ONTO, WE GET THAT $g \circ f$ IS A BIJECTION

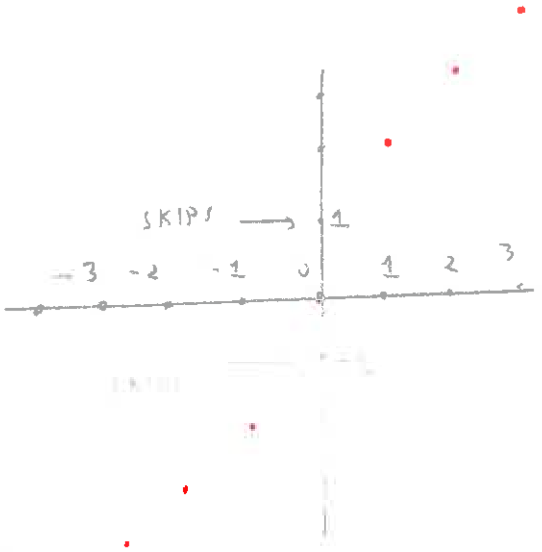
3) ONE-TO-ONE AND ONTO: LET $f(N) = N$ EXCEPT $f(1) = -1$ AND $f(-1) = 1$



$f : \mathbb{Z} \to \mathbb{Z}$

12 ~~3~~. Give an example of a function ~~$f : \mathbb{N} \to \mathbb{N}$~~ that is one-to-one but not onto; an example that is onto but not one-to-one; and and example that is one-to-one and onto but different from $f(n) = n$ or $f(n) = -n$. You don't have to come up with formulas, graphs are ok too.

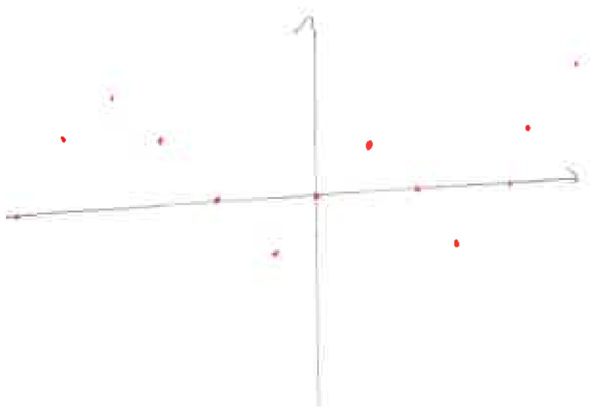
1) ONE-TO-ONE BUT NOT ONTO:



LET $f(N) = \begin{cases} N+1 & \text{IF } N \geq 1 \\ 0 & \text{IF } N = 0 \\ N-1 & \text{IF } N \leq -1 \end{cases}$

THEN f IS ONE-TO-ONE, BUT NOT ONTO, BECAUSE 1 (AND -1) ARE NOT IN THE RANGE OF f

2) ONTO BUT NOT ONE-TO-ONE



(THE SAWTOOTH EXAMPLE FROM THE HOMEWORK, LIKE

13. Define the relation $\sim$ on $A = \mathbb{R}$ by $x \sim y$ if and only if $x - y$ is rational. Show that $\sim$ is an equivalence relation.

$x \sim x$? IS $x - x$ RATIONAL? YES, BECAUSE $x - x = 0 \in \mathbb{Q}$

$x \sim y \Rightarrow y \sim x$? YES SUPPOSE $x - y = q$ WITH $q \in \mathbb{Q}$

THEN $y - x = -(x - y) = -q \in \mathbb{Q}$

SO $y - x$ IS RATIONAL

$x \sim y$ & $y \sim z \Rightarrow x \sim z$? YES SUPPOSE $x - y = q$ AND $y - z = r$ WITH $q, r \in \mathbb{Q}$

THEN $x - z = (x - y) + (y - z) = q + r \in \mathbb{Q}$

SO $x - z$ IS RATIONAL

HENCE $\sim$ IS AN EQUIVALENCE RELATION