SOLUTIONS

MATH 200 – MIDTERM 1 – PRACTICE QUESTIONS

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Here are some practice questions for Midterm 1. I would like to emphasize that you that the actual exam covers not just those practice questions, I'd highly recommend also looking at the homework problems as well as the problems covered during the review session. Also make sure to look at the midterm I gave last semester. Please refer to the study guide if you want to know exactly what to study.

Date: Friday, March 10, 2016.

1. (Note that there might not be a T/F portion on the exam) Label the following statements as **TRUE (T)** or **FALSE (F)**. Any correct answer gives you 2 points, and any incorrect answer gives you 0 points. You do **NOT** get points off for an incorrect answer, and you do **NOT** need to justify your answers.

F (a) If P and Q are propositions, then

$$\exists x\,(P(x) \wedge Q(x)) \equiv (\exists x\, P(x)) \wedge (\exists x\, Q(x))$$

F (b) If a, b, c are positive integers, then $a|bc \Rightarrow [(a|b) or (a|c)]$

T (c) If a and b are relatively prime, then $lcm(a, b) = ab$

F (d) For any real numbers x and y, $\lceil xy \rceil = \lceil x \rceil \lfloor y \rfloor$

T (e) There is a statement that is logically equivalent to its inverse

EXPLANATIONS (JUST FOR YOU)

F (a) LET $P(x)$ = "x is EVEN", $Q(x)$ = "x is ODD"

THEN $\exists x\,(P(x) \wedge Q(x))$ IS F (B/C THERE IS NO INTEGER THAT IS BOTH EVEN & ODD)

BUT $(\exists x\, P(x)) \wedge (\exists x\, Q(x))$ IS T

EX $x = 2$ EX $x = 1$

F (b) $4 \mid 12 = 6 \times 2$, BUT $4 \nmid 6$ AND $4 \nmid 2$

T (c) BY THE HW, WE HAVE $LCM(a,b)\, GCD(a,b) = ab$

IF a & b ARE REL PRIME, THEN $GCD(a,b) = 1$, SO WE GET $LCM(a,b) = ab$

F (d) $x = \frac{1}{2}$, $y = \frac{1}{2}$, THEN $\lceil xy \rceil = \lceil \frac{1}{4} \rceil = 1$, BUT $\lceil x \rceil \lfloor y \rfloor = \lceil \frac{1}{2} \rceil \lfloor \frac{1}{2} \rfloor$

$= 1 \times 0 = 0$

T (e) LET $P(x)$ = "$x \geq 2$" AND $Q(x)$ = "$2x \geq 4$"

THEN $P \Rightarrow Q$ IS TRUE (B/C $x \geq 2 \Rightarrow 2x \geq 4$)

BUT ALSO $\sim P \Rightarrow \sim Q$ IS TRUE (B/C $x < 2 \Rightarrow 2x < 4$

2. Write out a truth table for

$$(p \Rightarrow r) \Leftrightarrow (q \Rightarrow r)$$

p	q	r	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r) \Leftrightarrow (q \Rightarrow r)$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

3. Determine whether the following argument is valid

$$p \Rightarrow (q \vee \sim r)$$
$$p \Rightarrow p \wedge r$$
$$\therefore p \Rightarrow r$$

p	q	r	$p \Rightarrow (q \vee (\sim r))$	$p \Rightarrow (p \wedge r)$	$p \Rightarrow r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	T	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

PREMISES CONCLUSION

NOTICE THAT WHENEVER THE PREMISES ARE TRUE, THE CONCLUSION IS TRUE TOO (THE T'S ALIGN), HENCE THE ARGUMENT IS VALID.

4. Three little proofs Part 1: Show that $\sqrt{2}$ is irrational.

SUPPOSE $\sqrt{2}$ IS RATIONAL, THAT IS $\sqrt{2} = \frac{a}{b}$ FOR $a, b \in \mathbb{Z}$, $b \neq 0$

WLOG, WE MAY ASSUME THAT a & b HAVE NO FACTORS IN COMMON

NOW $\sqrt{2} = \frac{a}{b} \implies a = \sqrt{2}\, b \implies a^2 = 2\underbrace{b^2}_{=K \in \mathbb{Z}} = 2K$

HENCE a^2 IS EVEN, AND SO a IS EVEN, AND SO $a = 2L$ FOR SOME $L \in \mathbb{Z}$

BUT THEN $a^2 = 2b^2 \implies (2L)^2 = 2b^2 \implies 4L^2 = 2b^2$

$\implies b^2 = 2\underbrace{L^2}_{=L' \in \mathbb{Z}} = 2L'$

HENCE b^2 IS EVEN, AND SO b IS EVEN

BUT THEN a & b ARE BOTH EVEN AND HENCE HAVE A FACTOR IN COMMON (NAMELY 2). BUT WE ASSUMED THAT a & b HAVE NO FACTORS IN COMMON! $\Rightarrow\Leftarrow$

HENCE $\sqrt{2}$ IS IRRATIONAL.

5. Three little proofs Part 2: Show that there are infinitely many prime numbers.

SUPPOSE THERE ARE ONLY FINITELY MANY PRIME NUMBERS

$2, 3, 5, 7, 11, \ldots, p$, AND LET p BE THE LARGEST ONE

LET $N = (2 \times 3 \times 7 \times 11 \times \cdots \times p) + 1$

NOTICE THAT $N > p$, AND SINCE p IS THE LARGEST PRIME, N IS COMPOSITE, AND HENCE THERE IS $q > 1$ PRIME WITH $q \mid N$

BUT SINCE q IS PRIME, $q = 2$ OR 3 OR $\cdots$ OR p, AND IN PARTICULAR

$q \mid (2 \times 3 \times \cdots \times p)$

IN PARTICULAR, SINCE $q \mid N$ AND $q \mid (2 \times \cdots \times p)$, $q \mid (N - (2 \times \cdots \times p)) = 1$, SO $q \mid 1$ AND HENCE $q = \pm 1$. BUT THIS CONTRADICTS THE FACT THAT $q > 1$ $\Rightarrow\!\Leftarrow$

HENCE THERE ARE INFINITELY MANY PRIMES.

6. Three little proofs Part 3: Show that there is are irrational numbers x and y such that x^y is rational.

CONSIDER $\sqrt{2}^{\sqrt{2}}$

CASE 1 $\sqrt{2}^{\sqrt{2}}$ IS RATIONAL

THEN LETTING $X=\sqrt{2}$ AND $Y=\sqrt{2}$ (IRRATIONAL), WE GET $X^Y = \sqrt{2}^{\sqrt{2}}$ RATIONAL ✓

CASE 2 $\sqrt{2}^{\sqrt{2}}$ IS IRRATIONAL

THEN LET $X = \sqrt{2}^{\sqrt{2}}$ AND $Y=\sqrt{2}$ (IRRATIONAL), AND WE GET

$$X^Y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2})^2} = (\sqrt{2})^2 = 2 \text{ RATIONAL ✓}$$

IN BOTH CASES, WE FOUND X & Y IRRATIONAL W/ X^Y RATIONAL

7. Show that $(n+1)(n+5)(n+9)$ is always divisible by 3

(N IS AN INTEGER)

BY QUOTIENT-REMAINDER W/ $d = 3$,

WE HAVE $N = 3q + r$ FOR SOME $q, r \in \mathbb{Z}$ W/ $0 \leq r < 3$

BUT $0 \leq r < 3 \Rightarrow r = 0, 1, 2$

CASE 1 $r = 0$, THEN $N = 3q$

AND $(N+1)(N+5)(N+9) = (3q+1)(3q+5)(3q+9)$

$$= 3\underbrace{\left[(3q+1)(3q+5)(q+3)\right]}_{:= K \in \mathbb{Z}} = 3K \checkmark$$

CASE 2 $r = 1$, THEN $N = 3q+1$

AND $(N+1)(N+5)(N+9) = (3q+2)(3q+6)(3q+10)$

$$= 3\underbrace{\left[(3q+2)(q+2)(3q+10)\right]}_{= K \in \mathbb{Z}} = 3K \checkmark$$

CASE 3 $r = 2$, THEN $N = 3q+2$

AND $(N+1)(N+5)(N+9) = (3q+3)(3q+8)(3q+11)$

$$= 3\underbrace{\left[(q+1)(3q+8)(3q+11)\right]}_{:= K \in \mathbb{Z}} = 3K \checkmark$$

IN ALL CASES, $(N+1)(N+5)(N+9)$ IS DIVISIBLE BY 3

8. Show that positive integer n is divisible by 9 if and only if the sum of its digits are divisible by 9.

   Hint: You can write n as $n = a_0 + a_1 10 + a_2 10^2 + \cdots + a_m 10^m$ for some m and some integers a_i between 0 and 9.

NOTICE THAT :

$a_0 \equiv a_0 \ \underline{\text{MOD}}\ 9$

$10 \equiv 1 \ \underline{\text{MOD}}\ 9$, SO $a_1 \times 10 \equiv a_1 \times 1 \equiv a_1 \ \underline{\text{MOD}}\ 9$

$10^2 \equiv 1^2 \equiv 1 \ \underline{\text{MOD}}\ 9$, SO $a_2 \times 10^2 \equiv a_2 \ \underline{\text{MOD}}\ 9$

$\forall k \in \mathbb{N}, 10^k \equiv 1^k \equiv 1 \ \underline{\text{MOD}}\ 9$, SO $a_k \times 10^k \equiv a_k \ \underline{\text{MOD}}\ 9$

AND SO $N \equiv a_0 + a_1 10 + a_2 10^2 + \cdots + a_m 10^m$

$\equiv a_0 + a_1 + a_2 + \cdots + a_m \ \underline{\text{MOD}}\ 9$

AND SO $N \equiv 0 \ \underline{\text{MOD}}\ 9$ IFF $\underbrace{a_1 + \cdots + a_m}_{\text{SUM OF DIGITS OF } N} \ \underline{\text{MOD}}\ 9$

IN OTHER WORDS, N IS DIVISIBLE BY 9 IFF THE SUM OF THE DIGITS OF N IS DIVISIBLE BY 9.

9. Show that there are infinitely many composite numbers

THERE ARE MANY WAYS OF PROVING THIS; HERE ARE A COUPLE OF MY FAVORITES.

PROOF #1 NOTICE THAT THE NUMBERS $2^2, 2^3, 2^4, \ldots$ ARE COMPOSITE (B/C THEY ARE ALL DIVISIBLE BY 2)

ALSO THEY ARE ALL DISTINCT B/C $2^m = 2^n \Rightarrow 2^{m-n} = 1 \Rightarrow m = n$

HENCE WE GET AN INFINITE LIST OF COMPOSITE NUMBERS

PROOF #2 SUPPOSE THERE ARE ONLY FINITELY MANY COMPOSITE NUMBERS, AND LET M BE THE LARGEST ONE.

THEN $N := 2M$ IS $> M$ AND COMPOSITE (B/C N IS DIVISIBLE BY 2)

AND THIS CONTRADICTS THAT N IS THE LARGEST COMPOSITE # $\Rightarrow\Leftarrow$

PROOF #3 (COURTESY ALEX PARKHURST, FROM FALL 2016)

SUPPOSE THERE ARE FINITELY MANY COMPOSITE NUMBERS AND LET M BE THE LARGEST ONE.

SINCE M IS COMPOSITE, $M = rs$ FOR SOME $r, s > 1$

CONSIDER $N := (r+1)s$

THEN $r+1 > 1$ AND $s > 1$, SO N IS COMPOSITE

MOREOVER $N = (r+1)s > rs = M$ AND SO $N > M$

BUT THIS CONTRADICTS THE FACT THAT THERE ARE ONLY FINITELY MANY COMPOSITE #s $\Rightarrow\Leftarrow$

10. Show that if $x - \lfloor x \rfloor \geq \frac{1}{2}$, then $\lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$

LET $N = \lfloor X \rfloor$, THEN $\underline{N \leq X < N+1}$ (*)

WANT TO SHOW $\lfloor 2X \rfloor = 2\lfloor X \rfloor + 1$

THAT IS $\lfloor 2X \rfloor = 2N + 1$

THAT IS $2N+1 \leq 2X < 2N+2$

MULTIPLYING (*) BY 2, WE GET $\underline{2N \leq 2X < 2N+2}$ (1)

AND FROM $X - \lfloor X \rfloor \geq \frac{1}{2}$, WE GET $X - N \geq \frac{1}{2} \Rightarrow X \geq N + \frac{1}{2}$

$\Rightarrow \underline{2X \geq 2N+1}$ (2)

COMBINING (1) & (2) WE GET $2N+1 \leq 2X < 2N+2$, WHICH IS WHAT WE WANT.

11. Show that if n is an integer, then

$$\left\lfloor \frac{\lfloor \frac{n}{3} \rfloor}{3} \right\rfloor = \left\lfloor \frac{n}{9} \right\rfloor$$

FIRST TRY USE ER ON N WITH $d = 9$ (NINE)

YOU COULD DO THIS, BUT THEN YOU'LL HAVE NINE CASES

SECOND TRY LET $M = \lfloor \frac{N}{3} \rfloor$, THEN $M \leq \frac{N}{3} < M+1$

$3M \leq N < 3M+3 \quad (*)$

THEN BY ER ON M WITH $d = 3$, WE GET

$M = 3q + r$ WITH $q, r \in \mathbb{Z}$, $0 \leq r < 3 \Rightarrow r = 0, 1, 2$

CASE 1 $r = 0$ THEN $M = 3q$

AND $\left\lfloor \frac{\lfloor \frac{N}{3} \rfloor}{3} \right\rfloor = \left\lfloor \frac{M}{3} \right\rfloor = \left\lfloor \frac{3q}{3} \right\rfloor = q$

ALSO FROM $(*)$, WE GET $3M \leq N < 3M+3$

$9q \leq N < 9q+3$

$q \leq \frac{N}{9} < q + \frac{1}{3} < q+1 \Rightarrow q \leq \frac{N}{9} < q+1$

SO $\left\lfloor \frac{N}{9} \right\rfloor = q$ ✓

CASE 2 $r = 1$ THEN $M = 3q+1$

AND $\left\lfloor \frac{\lfloor \frac{N}{3} \rfloor}{3} \right\rfloor = \left\lfloor \frac{M}{3} \right\rfloor = \left\lfloor \frac{3q+1}{3} \right\rfloor = \lfloor \underbrace{q}_{\in \mathbb{Z}} + \frac{1}{3} \rfloor = q + \lfloor \frac{1}{3} \rfloor = q$

AND FROM $(*)$, WE GET $3M \leq N < 3M+3 \Rightarrow 9q+3 \leq N < 9q+6$

$\Rightarrow (q \leq)\ q + \frac{1}{3} \leq \frac{N}{9} < q + \frac{2}{3}\ (< q+1)$

$\Rightarrow q \leq \frac{N}{9} < q+1$

HENCE $\left\lfloor \frac{N}{9} \right\rfloor = q$ ✓

CASE 3 $r = 2$ SIMILAR (BUT DO IT!)

12. What is the remainder when 2^{345} is divided by 11?

BY FERMAT'S LITTLE THEOREM

$$2^{11} \equiv 2 \ \underline{\text{MOD}}\ 11 \quad \text{PRIME}$$

$$\cancel{2} \times 2^{10} \equiv \cancel{2} \times 1 \ \underline{\text{MOD}}\ 11 \qquad (\text{2 AND 11 ARE REL PRIME})$$

$$2^{10} \equiv 1 \ \underline{\text{MOD}}\ 11$$

NOW

$$\begin{array}{r|l} & 34 \\ \hline 10 & 345 \\ & 30 \\ \hline & 45 \\ & 40 \\ \hline & 5 \end{array}$$

(SILLY, I KNOW ☺)

$$345 = (34) \times 10 + 5$$

AND SO

$$2^{345} \equiv 2^{10 \times 34 + 5} \equiv (\underbrace{2^{10}}_{\equiv 1})^{34}\, 2^5 \equiv (1^{34})\, 2^5 \equiv 2^5 \ \underline{\text{MOD}}\ 11$$

BUT

$$2^5 = 32 = 2 \times 11 + 10$$

$$\begin{array}{r|l} & 2 \\ \hline 11 & 32 \\ & 22 \\ \hline & 10 \end{array}$$

AND SO $2^5 \equiv 10 \ \underline{\text{MOD}}\ 11$

HENCE $2^{345} \equiv 10 \ \underline{\text{MOD}}\ 11$ AND SINCE $0 \leq 10 < 11$, WE GET THAT THE REMAINDER IS 10.

13. Show that for all integers $n \geq 2$,

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$$

LET P_N BE THE PROPOSITION

$$\text{"} \sum_{i=1}^{N} i(i+1) = \frac{N(N-1)(N+1)}{3} \text{"}$$

<u>BASE CASE</u> <u>$N=2$</u> $\quad \sum_{i=1}^{1} i(i+1) = (1)(2) = 2$ AND $\frac{(2)(1)(3)}{3} = 2$

<u>IND STEP</u> SUPPOSE P_N IS TRUE, THAT IS $\sum_{i=1}^{N-1} i(i+1) = \frac{N(N-1)(N+1)}{3}$

SHOW P_{N+1} IS TRUE, THAT IS $\sum_{i=1}^{N} i(i+1) = \frac{(N+1)(N)(N+2)}{3}$

BUT $\sum_{i=1}^{N} i(i+1) = (1)(2) + (2)(3) + \cdots + (N)(N+1)$

$$= \underbrace{(1)(2) + (2)(3) + \cdots + (N-1)(N)} + (N)(N+1)$$

$$= \sum_{i=1}^{N-1} i(i+1) + (N)(N+1)$$

B/C P_N IS TRUE

$$= \frac{N(N-1)(N+1)}{3} + (N)(N+1)$$

$$= N(N+1)\left(\frac{N-1}{3} + 1\right) = N(N+1)\left(\frac{N-1+3}{3}\right)$$

$$= \frac{N(N+1)(N+2)}{3}$$

$$= \frac{(N+1)N(N+2)}{3} \checkmark$$

HENCE P_{N+1} IS TRUE, AND BY INDUCTION P_N IS TRUE $\forall N \geq 2$.

14. Show that for all integers $n \geq 0$,

$$2^n < (n+2)!$$

LET P_N BE THE PROPOSITION "$2^N < (N+2)!$"

BASE CASE $N=0$ THEN $2^0 = 1$ AND $(0+2)! = 2! = 2 > 1$

SO $2^0 < (0+2)!$ ✓

INDUCTIVE STEP SUPPOSE P_N IS TRUE, THAT IS $2^N < (N+2)!$

SHOW P_{N+1} IS TRUE, THAT IS $2^{N+1} < (N+3)!$

BUT $(N+3)! = (N+3)\,(N+2)!$

$> (N+3)\,2^N$ (BECAUSE P_N IS TRUE)

BUT SINCE $N \geq 0$, $N+3 \geq 3 > 2$

AND SO $(N+3)2^N > (2)(2^N) = 2^{N+1}$

AND SO WE GET $(N+3)! > (N+3)2^N > 2^{N+1}$

SO $(N+3)! > 2^{N+1}$

HENCE P_{N+1} IS TRUE, AND SO BY INDUCTION P_N IS TRUE $\forall N \geq 0$.

15. Suppose there are n people at a party and suppose that each shakes hands with all the other people present. Show that $\frac{n(n-1)}{2}$ handshakes must occur.

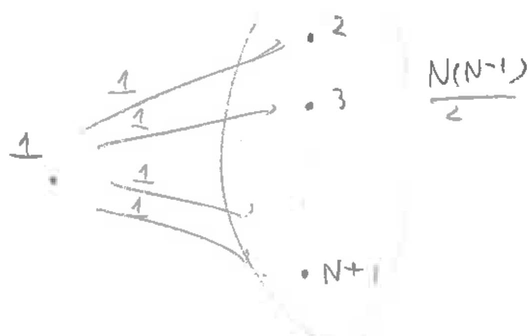
LET P_N BE THE PROP THAT IN A GROUP OF N PEOPLE, $\frac{N(N-1)}{2}$ HANDSHAKES OCCUR.

BASE CASE $N=1$ IF THERE'S JUST 1 PERSON, 0 HANDSHAKES OCCUR

AND $0 = \frac{(1)(0)}{2}$ ✓

INDUCTIVE STEP SUPPOSE P_N IS TRUE, THAT IS IN A GROUP OF N PEOPLE, $\frac{N(N-1)}{2}$ HANDSHAKES OCCUR

SHOW P_{N+1} IS TRUE, THAT IS IN A GROUP OF $N+1$ PEOPLE, $\frac{(N+1)N}{2}$ HANDSHAKES OCCUR



SUPPOSE THERE ARE $N+1$ PEOPLE IN THE ROOM.

THEN PERSON #1 SHAKES HANDS WITH PERSON $\underbrace{\#2, \#3, \dots, \#N+1}_{N \text{ PEOPLE}}$

AND THEN PEOPLE #2, 3, N+1 SHAKE HANDS W/ EACH OTHER

SO TOTAL # OF HANDSHAKES = (# OF HANDSHAKES OF 1 WITH $\underbrace{2, 3, \dots, N+1}_{N \text{ PEOPLE}}$)

+ (# OF HANDSHAKES OF $\underbrace{2, 3, \dots, N+1}_{N \text{ PEOPLE}}$ W/ EACH OTHER)

$$= N + \underbrace{\frac{N(N-1)}{2}}_{\text{BY THE INDUCTION HYP}} = N\left(1 + \left(\frac{N-1}{2}\right)\right) = N\left(\frac{2+N-1}{2}\right) = \frac{N(N+1)}{2} ✓$$

HENCE P_{N+1} IS TRUE, HENCE P_N IS TRUE $\forall N \geq 0$

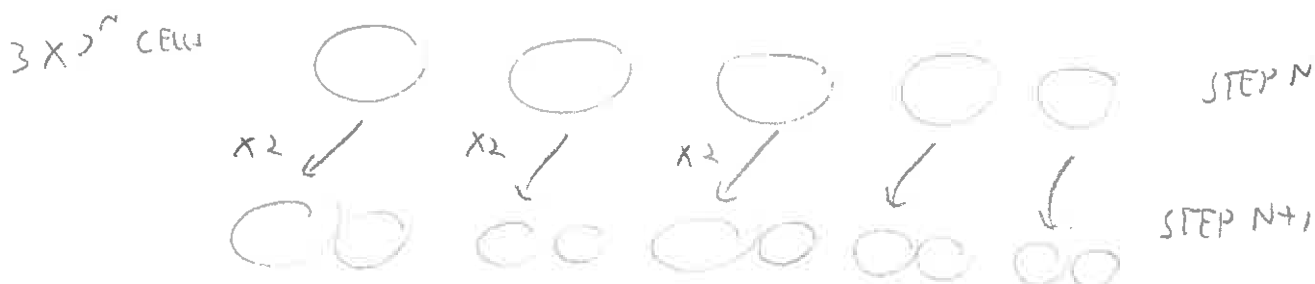
16. Suppose that at the beginning, there are 3 cells, and after each step, each cell splits into 2. Show by induction on n that after n steps, there are 3×2^n cells.

LET P_N BE THE PROPOSITION THAT AFTER N STEPS THERE ARE 3×2^N CELLS.

BASE CASE $N=0$ INITIALLY (AFTER 0 STEPS), THERE ARE $3 = 3 \times 2^0$ CELLS ✓

IND STEP SUPPOSE THAT P_N IS TRUE, THAT IS AFTER N STEPS THERE ARE 3×2^N CELLS.

SHOW THAT P_{N+1} IS TRUE, THAT IS AFTER N+1 STEPS, THERE ARE $3 \times 2^{N+1}$ CELLS.



B/C P_N IS TRUE, AFTER N STEPS, THERE ARE 3×2^N CELLS

BUT WE KNOW THAT AFTER EACH STEP, THE # OF CELLS DOUBLES

SO AFTER N+1 STEPS, THERE ARE $2 \times (\#\text{CELLS IN STEP } N)$

$$= 2 \times (3 \times 2^N)$$

$$= 3 \times 2 \times 2^N = 3 \times 2^{N+1} \text{ CELLS}$$

HENCE P_{N+1} IS TRUE, SO P_N IS TRUE $\forall N \geq 0$.