MATH 200 – FINAL EXAM – PRACTICE QUESTIONS

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Note: Here are some practice questions for the final exam. I would like to remind you that the actual exam covers not just those practice questions, but also everything covered in lecture, in the book, and the homework. Please refer to the study guide if you want to know exactly what to study. Also, don't forget to look at the practice questions for the first midterm and for the second midterm, since they are a source of great exam questions as well.

Date: Thursday, May 4, 2017.

1. Label the following statements as **TRUE (T)** or **FALSE (F)**. Any correct answer gives you 2 points, and any incorrect answer gives you 0 points. You do **NOT** get points off for an incorrect answer, and you do **NOT** need to justify your answers.

   (a) The relation $A \sim B$ if and only if $P(A) = P(B)$ is an equivalence relation on the set of events.

   (b) There are $\frac{9!}{4}$ different ways to arrange the letters in the word TABRIZIAN.

   (c) If you toss a fair coin 8 times, the probability of getting ≥ 4 heads is $\frac{1}{2}$.

   (d) You must pick at least 20 integers from 1 to 100 to be sure of getting one that is divisible by 5.

   (e) Suppose you toss a fair coin and you win $\$3$ if you get H and you win $\$1$ if you get T, and suppose it costs $\$2$ to play this game, then the expected value of your game is $\$0$.

2. Suppose there are two urns. Urn 1 contains two black balls (labeled B_1 and B_2) and one white ball. Urn 2 contains one black ball and two white balls (labeled W_1 and W_2). Suppose the following experiment is performed: One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.

   (a) What is the probability that two black balls are chosen?
   (b) What is the probability that two balls of opposite color are chosen?

3. A telephone number is formed using 7 digits (from 0 to 9). What is the probability that a randomly chosen seven-digit phone number would have at least one repeated digit?

4. What is the probability that

   (a) The top and bottom cards of a randomly shuffled deck of 52 cards are both aces?

   (b) A five-card poker hand contains the ace of hearts?

   (c) A five-card poker hand contains a full house (= 3 cards of the same rank and 2 cards of the same rank).

5. Suppose that one out of every 1000 Pokémon is shiny, and a Pokéball with the following properties:

   - If the Pokémon is shiny, the chances of catching it is 0.1 percent.
   - If the Pokémon is not shiny, the chances of catching it is 50 percent.

   Prof. Oak tells you that there is a magic potion that doubles the HP of your Pokémon, but the only way to get it is by winning the following game: Suppose you toss a crooked coin with the probability of having H is the probability of a Pokémon being shiny given that you catch it. You win if, out of 10 tosses, you get exactly 3 heads.

   What is the probability of getting the magic potion?

6. The following 'paradox' (although it's not really a paradox) is called "The Gambler's Ruin."

   Suppose a gambler repeatedly bets on the following game: If the coin comes up H, the gambler wins $\$1$, but if it comes up T, the gambler loses $\$1$. The gambler will quit playing either when he is ruined (has $\$0$) or when he has $\$100$.

   Let P_n be the probability that he will become broke when he begins playing with $\$n$.

   (a) Justify that we obtain the difference equation
   $$P_n = \frac{1}{2}P_{n+1} + \frac{1}{2}P_{n-1}$$
   (here $1 \leq n \leq 99$).

   (b) Solve this difference equation. Notice that you actually know what P_0 and P_{100} are!

   (c) What are the chances of going broke when the gambler plays with $\$20$? With $\$50$? with $\$99$? Notice that the closer n is to 100, the closer P_n is to 0. That is, the more modest the gambler is in his goal, the more likely he is to reach it.

7. Pascal's formula states that for n and r positive integers and $r \leq n$, we have

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Prove Pascal's formula both using an algebraic argument (= a calculation) and a combinatorial argument (using counting).

Hint: For the combinatorial argument, suppose you have a set of $n+1$ elements, say $A = \{1, 2, \cdots, n+1\}$ and suppose you want to choose a subset B of A with r elements. Argue in terms of whether 1 is in B or not.

8. Show that

$$\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{4}\binom{n}{2}+\cdots+(-1)^{n-1}\frac{1}{2^{n-1}}\binom{n}{n-1}=\begin{cases}0 & \text{if } n \text{ is even}\\ \frac{1}{2^{n-1}} & \text{if } n \text{ is odd}\end{cases}$$

9. Suppose Dunkin' Donuts offers 6 different kinds of donuts: Chocolate, Cinnamon, Powdered Sugar, Boston Cream, Jelly, and Apple Cider. Today they only have 10 Chocolate donuts left but 40 each of the other kinds of donuts.

   (a) How many different selections of 20 donuts are there?

   (b) Suppose in addition to only having 10 Chocolate donuts, it only has 8 Powdered Sugar Donuts. How many different selections of 20 donuts are there?

10. Show that if A and B are **finite** and **of the same size**, then $f : A \to B$ is one-to-one if and only if it is onto. Is this still true if A and B are infinite (but of the same size)?

11. In the following, either draw a graph with the specified properties, or explain why no such graph exists:

(a) A simple graph with 4 vertices of degrees 1, 2, 3, and 4.

(b) A simple graph with 6 edges and all vertices of degree 3.

(c) A graph with 4 vertices of degrees 1, 2, 3, and 3.

(d) A circuit-free graph with nine vertices and six edges

(e) A tree with nine edges and nine vertices

(f) A tree with five vertices and total degree 10

(g) A full binary tree with 12 vertices

(h) A binary tree with 9 vertices

(i) A full binary tree with 8 internal vertices and 7 terminal vertices

(j) A binary tree with height 4 and 8 terminal vertices

12. (a) Find an Euler Circuit in the graphs of exercises 16 and 17 on page 658, or show that such a circuit doesn't exist

    (b) Find a Hamiltonian circuit in the graph of exercise 24 on page 659.

13. (a) Is the following a tautology? Why or why not?

$$[(p \vee q) \wedge ((\sim p) \vee r)] \Rightarrow (q \vee r)$$

(b) Are the following two statements logically equivalent?

$$(\exists x P(x)) \wedge (\exists x Q(x)), \qquad \exists x (P(x) \wedge Q(x))$$

14. Show that if a number of the form $2^n - 1$ is prime, then n must be prime as well.

Hint: Use the identity

$$a^m - b^m = (a - b)(a^{m-1}b + a^{m-2}b^2 + \cdots + a^2b^{m-2} + ab^{m-1})$$

15. Show that for all positive integers m and n, we have

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor$$

Hint: Use the quotient-remainder theorem with $d = m$. Treat the case $r = 0$ separately (using that the ceiling of an integer is the integer itself).

16. (a) Show that for all $n \geq 0$, we have

$$\frac{1}{\sqrt{n+1}} \geq 2(\sqrt{n+2} - \sqrt{n+1})$$

Hint: Multiply both sides by $\sqrt{n+2} + \sqrt{n+1}$

(b) Prove by induction on n that for all positive integers n, we have

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq 2(\sqrt{n+1} - 1)$$

Cultural note: This inequality shows that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

17. Use induction to prove that any integer $n \geq 1$ can be written as $n = 2^k m$ for some positive and odd integer m and some non-negative integer k.

    Hint: Argue in terms of whether n is even or n is odd.

18. Consider the following variation of the Tower of Hanoi Problem: There are 3 poles in a row, and $\frac{n(n+1)}{2}$ disks, one disk of size 1, two of size 2, $\cdots$ and n of size n, where n is any positive integer. Initially one of the poles contains all the disks placed on top of each other in sets of decreasing size (so at the bottom there are the n plates of size n, then the $n-1$ plates of size $n-1$, $\cdots$, and at the top there is 1 plate of size 1). Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be plaed on top of one of the same size. Let s_n be the minimum number of moves needed to transfer a tower of $\frac{n(n+1)}{2}$ disks from one pole to another.

    Find a recurrence relation for s_n

19. Define the symmetric difference of two sets by $A\Delta B = (A - B) \cup (B - A)$. Prove or disprove the following statements:

(a)
$$(A\Delta B)\Delta A = A$$

(b)
$$(A\Delta B)^c = A \cap B$$

20. Show that the set of all real numbers x with the property that there exist integers $a \neq 0, b, c$ such that $ax^2 + bx + c = 0$ is countable.

21. Define the following relation $\sim$ on $\mathbb{R}^2$:

$$(x, y) \sim (z, t) \iff x^2 + y^2 = z^2 + t^2$$

(a) Show that $\sim$ is an equivalence relation.

(b) What do the equivalence classes of $\sim$ look like?

22. Define the relation $\sim$ on $A = \mathbb{R}$ by $x \sim y$ if and only if $x - y$ is rational.

    (a) Show that $\sim$ is an equivalence relation

    (b) Show that for every x, $[x]$ (the equivalence class of x) is countable
    Hint: Show that $f : \mathbb{Q} \longrightarrow [x]$ defined by $f(q) = x + q$ is a bijection.

    (c) Show that there are uncountably many *distinct* equivalence classes $[x]$, where $x \in \mathbb{R}$.