SOLUTIONS

MATH 200 – FINAL EXAM – PRACTICE QUESTIONS

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Note: Here are some practice questions for the final exam. I would like to remind you that the actual exam covers not just those practice questions, but also everything covered in lecture, in the book, and the homework. Please refer to the study guide if you want to know exactly what to study. Also, don't forget to look at the practice questions for the first midterm and for the second midterm, since they are a source of great exam questions as well.

Date: Thursday, May 4, 2017.

1. Label the following statements as **TRUE (T)** or **FALSE (F)**. Any correct answer gives you 2 points, and any incorrect answer gives you 0 points. You do **NOT** get points off for an incorrect answer, and you do **NOT** need to justify your answers.

   Ⓣ (a) The relation $A \sim B$ if and only if $P(A) = P(B)$ is an equivalence relation on the set of events.

   Ⓣ (b) There are $\frac{9!}{4}$ different ways to arrange the letters in the word TABRIZIAN.

   Ⓕ (c) If you toss a fair coin 8 times, the probability of getting ≥ 4 heads is $\frac{1}{2}$.

   Ⓕ (d) You must pick at least 20 integers from 1 to 100 to be sure of getting one that is divisible by 5.

   Ⓣ (e) Suppose you toss a fair coin and you win \$3 if you get H and you win \$1 if you get T, and suppose it costs \$2 to play this game, then the expected value of your game is \$0.

EXPLANATIONS (OPTIONAL)

(a) Ⓣ $A \sim A$ B/C $P(A) = P(A)$ ✓

$A \sim B \Rightarrow B \sim A$ B/C $P(A) = P(B) \Rightarrow P(B) = P(A)$ ✓

$A \sim B$ & $B \sim C \Rightarrow A \sim C$ B/C $P(A) = P(B)$ & $P(B) = P(C) \Rightarrow P(A) = P(C)$

(b) Ⓣ PLACE THE A: $\binom{9}{2}$

PLACE THE I: $\binom{7}{2}$

PLACE THE OTHER LETTERS $= 5!$

MULT RULE $\binom{9}{2}\binom{7}{2}5!$

$$= \frac{9\times 8\times 7\times 6\times 5!}{2!\;2!} = \frac{9!}{2!\,2!} = \frac{9!}{4}$$

(c) Ⓕ $P(\geq 4\ H) = 1 - P(< 4H) = 1 - P(\leq 3H)$

BUT $P(\leq 3\ H) = P(\text{NO HEAD}) + P(1\text{ HEAD}) + P(2\text{ HEADS}) + P(3\text{ HEADS})$

$$= \binom{8}{0}\left(\frac{1}{2}\right)^8 + \binom{8}{1}\left(\frac{1}{2}\right)^8 + \binom{8}{2}\left(\frac{1}{2}\right)^8 + \binom{8}{3}\left(\frac{1}{2}\right)^8$$

$$= \frac{1}{2^8}\left(1 + 8 + \frac{8\times 7}{2} + \frac{8\times 7\times 6}{\cancel{6}}\right) = \frac{1}{2^8}(1 + 8 + 28 + 56) = \frac{93}{256}$$

SO $P(\geq 4\ H) = 1 - \frac{93}{256} = \frac{163}{256} > \frac{128}{256} = \frac{1}{2}$

(d) Ⓕ NO, YOU NEED 81

(e) Ⓣ OUTCOMES $= (3-2)$ OR $(1-2) = 1$ OR -1

PROBABILITIES $= \frac{1}{2}$, $E = \frac{1}{2}\times 1 + \frac{1}{2}\times(-1) = 0$

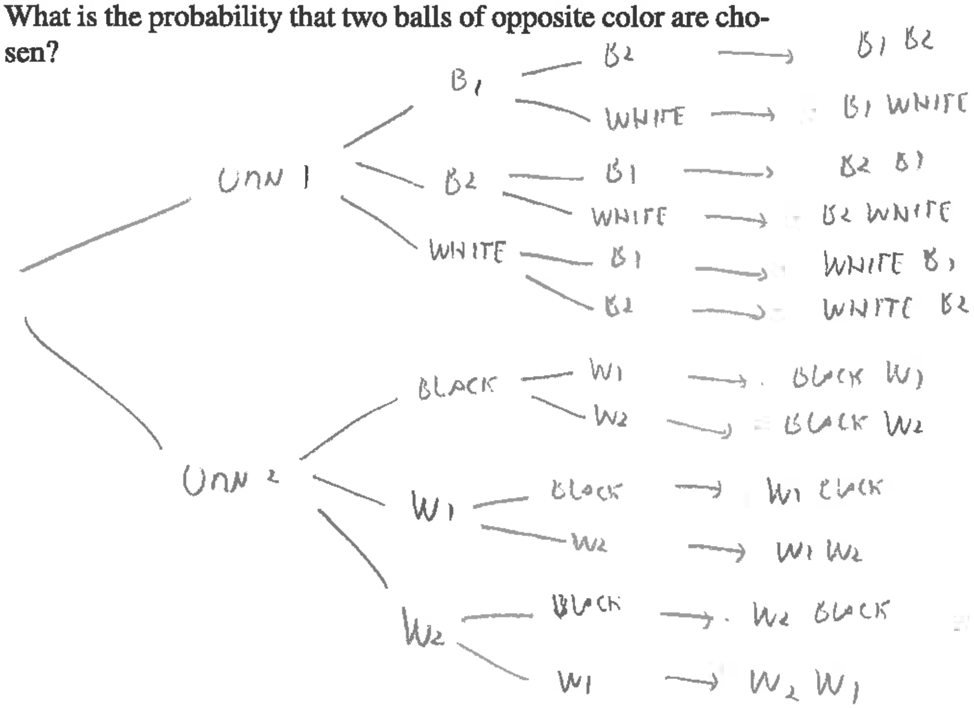
[redacted] [redacted]

2.[redacted] Suppose there are two urns. Urn 1 contains two black balls (labeled B_1 and B_2) and one white ball. Urn 2 contains one black ball and two white balls (labeled W_1 and W_2). Suppose the following experiment is performed: One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.

(a) What is the probability that two black balls are chosen?

(b) What is the probability that two balls of opposite color are chosen?

POSSIBILITIES



TOTAL # OF CHOICES = 12

(a) FAVORABLE OUTCOMES = 2 (B1 B2 AND B2 B1)

PROBABILITY = $\frac{2}{12} = \boxed{\frac{1}{6}}$

(b) FAVORABLE OUTCOMES = 8 (EVERYTHING EXCEPT B1 B2, B2 B1, W1 W2, W2 W1)

PROBABILITY = $\frac{8}{12} = \boxed{\frac{2}{3}}$

FROM 0, 1, 2, ..., 9

3 · [redacted] **A telephone number is formed using 7 digits. What is the probability that a randomly chosen seven-digit phone number would have at least one repeated digit?**

1) LET'S FIND THE PROBABILITY OF THE EVENT THAT THERE ARE <u>NO</u> REPEATED DIGITS :

$$\underset{\text{10 CHOICES}}{\underline{10}}\quad \underset{\text{9 CHOICES}}{\underline{9}}\quad \underline{8}\quad \underline{7}\quad \underline{6}\quad \underline{5}\quad \underline{4}$$

FAVORABLE OUTCOMES = # 7 − PERMUTATIONS OF 10 OBJECTS

$= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$

TOTAL OUTCOMES = $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$ (BY MULTIPLICATION RULE)

$= 10^7$

$$\text{PROBABILITY} = \frac{\cancel{10} \times 9 \times \cancel{8} \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4}}{\cancel{10} \times \cancel{10} \times \underset{5}{\cancel{10}} \times \underset{5}{\cancel{10}} \times \underset{5}{\cancel{10}} \times \underset{5}{\cancel{10}} \times \underset{5}{\cancel{10}}}$$

$$= \frac{9 \times 7 \times 3}{5 \times 5 \times 5 \times 5 \times 5} = \frac{9 \times 7 \times 3}{5^5} = \frac{3^3 \times 7}{5^5}$$

OK TO LEAVE IT LIKE THAT →

2) NOW BY THE FORMULA OF THE PROBABILITY OF THE COMPLEMENT

PROBABILITY OF AT <u>LEAST 1</u> REPEATED DIGIT

$= 1 - P(\text{NO REPEATED DIGIT})$

$$\boxed{= 1 - \frac{3^3 \times 7}{5^5}} \quad \leftarrow \text{OK TO LEAVE IT LIKE THAT}$$

4. What is the probability that

(a) The top and bottom cards of a randomly shuffled deck of 52 cards are both aces?

(b) A five-card poker hand contains the ace of hearts?

(c) A five-card poker hand contains a full house (= 3 cards of the same ~~suits~~ and 2 cards of the same ~~suit~~, but not all of the same suit) RANKS RANK

(a) WE REALLY ONLY CARE ABOUT THE TOP & BOTTOM CARDS,

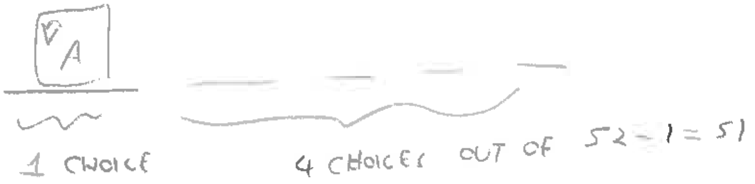
SO TOTAL OUTCOMES = $\binom{52}{2}$

FAVE OUTCOMES = $\binom{4}{2}$ ← CHOOSING 2 ACES FROM 4 ACES

$$P = \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{\frac{4\times3}{\cancel{2}}}{\frac{52\times51}{\cancel{2}}} = \frac{\cancel{4}\times\cancel{3}}{\underset{13}{\cancel{52}}\times\underset{17}{\cancel{51}}} = \frac{1}{13\times17} = \boxed{\frac{1}{221}}$$

(b) TOTAL OUTCOMES = $\binom{52}{5}$

FAVE OUTCOMES REMEMBER THE ORDER DOESN'T MATTER



$$= 1 \times \binom{51}{4} = \binom{51}{4}$$

$$\boxed{P = \frac{\binom{51}{4}}{\binom{52}{5}}}$$

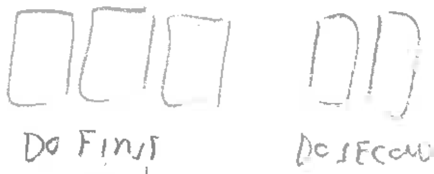
STEP 1 DENOM OF TRIPLE $\binom{13}{1}$

STEP 2 SUIT FOR TRIPLE $\binom{4}{3}$

STEP 3 DENOM OF PAIR $\binom{12}{1}$

STEP 4 SUIT FOR PAIR $\binom{4}{2}$

FAVE = $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$



(c) TOTAL = $\binom{52}{5}$

$$P = \left(\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}\right) \Big/ \binom{52}{5}$$

5. Suppose that one out of every 1000 Pokémon is shiny, and a Pokéball with the following properties:

- If the Pokémon is shiny, the chances of catching it is 0.1 percent.
- If the Pokémon is not shiny, then chances of catching it is 50 percent.

Prof. Oak tells you that there is a magic potion that doubles the HP of your Pokémon, but the only way to get it is by winning the following game: Suppose you toss a crooked coin with the probability of having H is the probability of a Pokémon being shiny given that you catch it. You win if, out of 10 tosses, you get exactly 3 heads.

What is the probability of getting the magic potion?

FIRST, LET'S FIGURE OUT $P(\text{SHINY} \mid \text{CATCH IT})$

BY BAYES: $P(\text{SHINY} \mid \text{CATCH IT}) = \dfrac{P(\text{CATCH IT} \mid \text{SHINY})P(\text{SHINY})}{P(\text{CATCH IT} \mid \text{SHINY})P(\text{SHINY}) + P(\text{CATCH} \mid \text{NOT S})P(\text{NOT}}$

$$= \frac{(0.001)(0.001)}{(0.001)(0.001) + (0.5)(0.999)} \qquad (\approx 2 \cdot 10^{-6}) \leftarrow \text{CALL THIS } p$$

THEN $P(\text{EXACTLY } 3\text{ H}) = \binom{10}{3} p^{3} (1-p)^{7}$

↳ WHY? BY INDEPENDENCE, THE ORDER DOESN'T MATTER, SO WE ASK OURSELVE

IN HOW MANY WAYS CAN WE GET THE COMBINATION

HHH TTT TTTT ? $\binom{10}{3}$

AND THE COMBINATION HHH TTTTTTT HAS PROBABILITY

$$P(\text{HHH TTT TTT}) = P(H)P(H)P(H)\,P(T)P(T)P(T)P(T)P(T)P(T)P(T$$

$$= ppp(1-p)(1-p)(1-p)(1-p)(1-p)(1-p)(1-p)$$

$$= p^{3}(1-p)^{7}$$

6. The following 'paradox' (although it's not really a paradox) is called "The Gambler's Ruin."

Suppose a gambler repeatedly bets on the following game: If the coin comes up H, the gambler wins \$1, but if it comes up T, the gambler loses \$1. The gambler will quit playing either when he is ruined (has \$0) or when he has \$100.

Let P_n be the probability that he will become broke when he begins playing with $\$n$.

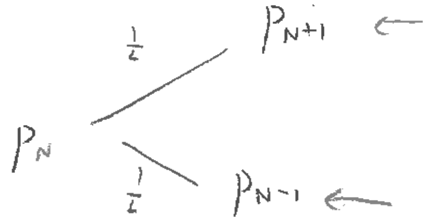
(a) Justify that we obtain the difference equation

[redacted] $P_N = \frac{1}{2} P_{N+1} + \frac{1}{2} P_{N-1}$

(b) Solve this difference equation. Notice that you actually know what P_0 and P_{100} are!

(c) What are the chances of going broke when the gambler plays with \$20? With \$50? with \$99? Notice that the closer n is to 100, the closer P_n is to 0. That is, the more modest the gambler is in his goal, the more likely he is to reach it.

(a) SUPPOSE THE PLAYER STARTS W/ \$ N
THEN THE PROB OF BEING BROKE IS P_N



BUT IF HE PLAYS AND WINS, HE HAS \$ N+1, SO THE PROBABILITY OF NOT GETTING BROKE IS P_{N+1}

IF HE LOSES, HE HAS \$ N-1, SO GET P_{N-}

HENCE P_N = P(BROKE W/ N\$)

= P(BROKE W/ N\$ ∩ WINNING) + P(BROKE W/ N\$ ∩ LOSING)

= P(BROKE W/ N\$ | WINNING) P(WINNING) + P(BROKE W/ N\$ | LOSING) P(LOSING)

$= P_{N+1}\left(\frac{1}{2}\right) + P_{N-1}\left(\frac{1}{2}\right)$ ✓

(b) (TURN PAGE)

PROBLEM 6 (CONTINUED)

(b) WE CAN WRITE THIS AS $P_{N+1} - 2P_N + P_{N-1} = 0$

(OR $P_{N+2} - 2P_{N+1} + P_N = 0$ IF YOU WISH)

AUX: $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$ REPEATED ROOT

$$P_N = A\,1^N + B\underline{N}\,1^N = A + BN$$

NOW $P_0 = 1$ (IF HE STARTS AT \$0, THEN FOR SURE HE'S ALREADY BROKE)

AND $P_{100} = 0$ (IF HE STARTS AT \$100, THEN HE QUITS SO NEVER GOES BROKE)

$$P_0 = 1 \Rightarrow A + B(0) = 1 \Rightarrow A = 1 \Rightarrow P_N = 1 + BN$$

$$P_{100} = 0 \Rightarrow 1 + 100B = 0 \Rightarrow B = -\frac{1}{100}$$

SO $\boxed{P_N = 1 - \frac{N}{100}}$

(c) $P_{20} = 1 - \frac{20}{100} = 1 - \frac{1}{5} = \boxed{\frac{4}{5}}$

$$P_{50} = 1 - \frac{50}{100} = \boxed{\frac{1}{2}}$$

$$P_{99} = 1 - \frac{99}{100} = \boxed{\frac{1}{100}}$$

7 ~~6~~. Pascal's formula states that for n and r positive integers and $r \leq n$, we have

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Prove Pascal's formula both using an algebraic argument (= a calculation) and a combinatorial argument (using counting).

Hint: For the combinatorial argument, suppose you have a set of $n+1$ elements, say $A = \{1, 2, \cdots, n+1\}$ and suppose you want to choose a subset B of A with r elements. Argue in terms of whether 1 is in B or not.

ALGEBRAIC ARGUMENT

$$\binom{N}{r-1} + \binom{N}{r} = \frac{N!}{(r-1)!\,(N-r+1)!} + \frac{N!}{r!\,(N-r)!}$$

PUT UNDER COMMON DENOM

$$= \frac{N!\;r}{\underbrace{(r-1)!\,r}_{r!}\,(N-r+1)!} + \frac{N!\;(N-r+1)}{r!\,\underbrace{(N-r)!\,(N-r+1)}_{(N-r+1)!}}$$

$$= \frac{N!\,r + N!\,(N-r+1)}{r!\,(N-r+1)!}$$

NUM $= N!\,(r+N-r+1) = N!\,(N+1) = (N+1)!$

DENOM $= r!\,(N-r+1)! = r!\,(N+1-r)!$

SO $= \dfrac{(N+1)!}{r!\,(N+1-r)!} = \dbinom{N+1}{r}$

(TURN PAGE FOR COMBI. ARGUMENT)

PROBLEM 7 (CONTINUED)

COMBINATORIAL ARGUMENT

NOTICE $\binom{N+1}{r}$ COUNTS THE # OF SUBSETS W/ r ELEMENTS OF A SET A WITH $N+1$ ELEMENTS

LET $A = \{1, 2, \dots, N+1\}$ AND B BE AN r-ELEMENT SUBSET. THEN $1 \in B$ OR $1 \notin B$

CASE 1 IF $1 \in B$, THEN $B = \{1, b_1, \dots, b_{r-1}\}$, WHERE

$$\underbrace{\{b_1, \dots, b_{r-1}\}}_{r-1 \text{ ELEMENTS}} \subset \underbrace{\{2, 3, \dots, N+1\}}_{N \text{ ELEMENTS}}$$

SO TO GET B, WE NEED TO COUNT THE # OF $r-1$ ELEMENT SUBSETS OF A SET OF N ELEMENTS; THERE ARE $\binom{N}{r-1}$ SUCH SUBSETS.

CASE 2 IF $1 \notin B$, THEN $B = \{b_1, \dots, b_r\}$

WHERE $\underbrace{\{b_1, \dots, b_r\}}_{r \text{ ELEMENTS}} \subset \underbrace{\{2, \dots, N+1\}}_{N \text{ ELEMENTS}}$,

SO HERE WE NEED TO COUNT THE # OF r ELEMENT SUBSETS FROM A SET OF N ELEMENTS, WHICH IS $\binom{N}{r}$.

CONCLUSION SINCE CASES 1 & 2 ARE DISJOINT

WE GET:

$$\underbrace{\#\ r\text{-ELEMENT SUBSETS OF } A}_{\binom{N+1}{r}} = \underbrace{\#\ r\text{-ELEMENT SUBSETS W/ } 1}_{\binom{N}{r-1}} + \underbrace{\#\ r \text{ ELEMENT SUBSETS W/O } 1}_{\binom{N}{r}}$$ ✓

8. Show that

$$\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{4}\binom{n}{2}+\cdots+(-1)^{n-1}\frac{1}{2^{n-1}}\binom{n}{n-1}=\begin{cases}0 & \text{if } n \text{ is even}\\ \frac{1}{2^{n-1}} & \text{if } n \text{ is odd}\end{cases}$$

NOTICE THIS SUM EQUALS TO

$$\binom{N}{0}\left(-\frac{1}{2}\right)^{0}1^{N}+\binom{N}{1}\left(-\frac{1}{2}\right)^{1}1^{N-1}+\cdots+\binom{N}{N-1}\left(-\frac{1}{2}\right)^{N-1}1^{1}+\binom{N}{N}\left(-\frac{1}{2}\right)1$$

$$-\binom{N}{N}\left(-\frac{1}{2}\right)^{N}1^{0}$$

$$=\left(-\frac{1}{2}+1\right)^{N}-\underbrace{\binom{N}{N}}_{1}\left(-\frac{1}{2}\right)^{N}\underbrace{1^{0}}_{1}$$ (BY BINOMIAL THEOREM)

$$=\left(\frac{1}{2}\right)^{N}-\left(-\frac{1}{2}\right)^{N}$$

NOW IF N IS EVEN, $N=2K$, THIS BECOMES $\left(\frac{1}{2}\right)^{2K}-\left(-\frac{1}{2}\right)^{2K}$

$$=\left(\frac{1}{4}\right)^{K}-\left(\frac{1}{4}\right)^{K}=0$$

IF N IS ODD, $N=2K+1$, THIS BECOMES

$$\left(\frac{1}{2}\right)^{2K+1}-\left(-\frac{1}{2}\right)^{2K+1}=\frac{1}{2^{2K+1}}-\underbrace{(-1)^{2K+1}}_{-1}\left(\frac{1}{2^{2K+1}}\right)$$

$$=\frac{1}{2^{2K+1}}+\frac{1}{2^{2K+1}}=\frac{2}{2^{2K+1}}=\frac{1}{\underbrace{2^{2K}}_{N=2K+1}}=\frac{1}{2^{N-1}}\ ✓$$

$$= \binom{25}{20} - \binom{14}{9} - \binom{16}{11} + \binom{5}{0}$$

— 5 BARS No X

$(S_{\leq 8})^c \cap (C_{\leq 10})^c$

$|S_{\geq 9} \cap C_{\geq 11}|$

5 BARS 0 X

5 BARS 11 X

$$= \binom{25}{20} - \binom{14}{9} - \binom{16}{11} + 1$$

9. Suppose Dunkin' Donuts offers 6 different kinds of donuts: Chocolate, Cinnamon, Powdered Sugar, Boston Cream, Jelly, and Apple Cider. Today they only have 10 Chocolate donuts left but 40 each of the other kinds of donuts.

(a) How many different selections of 20 donuts are there?

(b) Suppose in addition to only having 10 Chocolate donuts, it only has 8 Powdered Sugar Donuts. How many different selections of 20 donuts are there?

(a) WE WANT TO FIND $|C_{\leq 10}|$ = # SELECTIONS W/ ≤ 10 CHOCOLATE DONUTS

BY THE COMPLEMENT RULE, LET'S CALCULATE TOTAL − $|C_{\geq 11}|$

SELECTIONS W/ ≥ 11 CHOCOLATE DONUTS

IN OTHER WORDS C C C C C C C C C C C _ _ _ _ _ _ _ _ _

BOX & BARS

CH CI S BC J A

x x | x | x x | x | x | x x

SELECTION OF 9 DONUTS

5 BARS, 9 X

NAMELY, $|C_{\geq 11}| = \binom{14}{4}$

AND FOR TOTAL

CH CI S BC J A

x x | xxxx | x x | x x | x | x x x x

5 BARS 20 X

TOTAL = $\binom{25}{20}$, SO $|C_{\leq 10}| = \binom{25}{20} - \binom{14}{9}$

(b) BY THE COMPLEMENT RULE,

$$|C_{\leq 10} \cap S_{\leq 8}| = \text{TOTAL} - |(C_{\leq 10} \cap S_{\leq 8})^c|$$

$$= \binom{25}{20} - |C_{\geq 11} \cup S_{\geq 9}| \quad \text{(BY DE MORGAN)}$$

$$= \binom{25}{20} - |C_{\geq 11}| - |S_{\geq 9}| + |C_{\geq 11} \cap S_{\geq 9}| \quad \text{(INCLUSION-EXCLUSION)}$$

10. Show that if A and B are **finite and of the same size,** then $f : A \to B$ is one-to-one if and only if it is onto. Is this still true if A and B are infinite (but of the same size)?

FOR SIMPLICITY, LET'S ASSUME $A = B = \{1, \ldots, N\}$ (THE GENERAL CASE IS SIMILAR)

f 1-1 $\Rightarrow$ f ONTO

IF f IS NOT ONTO, THEN RAN(f) HAS STRICTLY LESS THAN N ELEMENTS.

BUT THEN YOU'RE ASSIGNING THE N INPUTS $1, \ldots, N$ (= PIGEONS) TO FEWER THAN N OUTPUTS $f(1), \ldots, f(N)$ (= HOLES), SO BY PG., THERE ARE 2 $\neq$ INPUTS $i \neq j$ W/ THE SAME OUTPUT $f(i) = f(j)$. SO f IS NOT 1-1

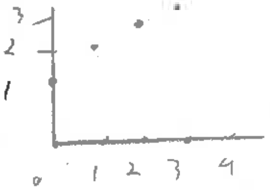
f ONTO $\Rightarrow$ f 1-1

IF f IS NOT 1-1, THEN THERE MUST BE $i \neq j$ W/ $f(i) = f(j)$

BUT THEN RAN(f) $= \{f(1), \ldots, f(N)\}$ HAS $< N$ ELEMENTS (B/C $f(i) \neq f(j)$) WHICH HAS N ELEMENTS $\{1, \ldots, N\}$,

AND HENCE CANNOT BE EQUAL TO $\{1, \ldots, N\}$,

HENCE f IS NOT ONTO $\{1, \ldots, N\}$.



COUNTEREXAMPLE $A = B = \mathbb{N}$, $f(N) = N+1$

THEN f IS 1-1 B/C $f(N) = f(M) \Rightarrow N+1 = M+1 \Rightarrow N = M$

f IS NOT ONTO ~~$\mathbb{N}$~~ B/C EVEN THOUGH $0 \in \mathbb{N}$ THERE IS NO $N \in \mathbb{N}$ SUCH THAT $f(N) = \underbrace{N+1}_{>0} = 0$

(f) DOES NOT EXIST. A TREE WITH 5 VERTICES MUST HAVE 4 EDGES AND HENCE TOTAL DEGREE 8 (BY THM 10.1.1)

(g) DOES NOT EXIST. A FULL BINARY TREE MUST HAVE AN ODD # OF VERTICES (THEOREM 10.6.1)

(h) 

(i) DOES NOT EXIST: A FULL BINARY TREE W/ 8 INTERNA VERTICES MUST HAVE 9 TERMINAL VERTICES (THM 10.6.1)

11. In the following, either draw a graph with the specified properties, or explain why no such graph exists:

(a) A simple graph with 4 vertices of degrees 1, 2, 3, and 4.

(b) A simple graph with 6 edges and all vertices of degree 3.

(c) A graph with 4 vertices of degrees 1, 2, 3, and 3.

(d) A circuit-free graph with nine vertices and six edges

(e) (Section 10.5) A tree with nine edges and nine vertices

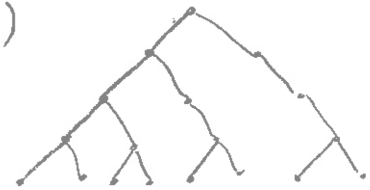
(f) (Section 10.5) A tree with five vertices and total degree 10

(g) (Section 10.6) A full binary tree with 12 vertices

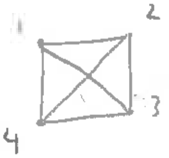
(h) (Section 10.6) A binary tree with 9 vertices

(i) (Section 10.6) A full binary tree with 8 internal ~~verties~~ VERTICES and 7 terminal vertices

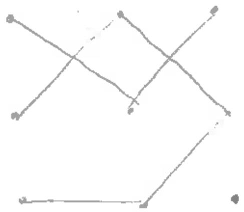
(j) (Section 10.6) A binary tree with height 4 and 8 terminal vertices

(j) 

(a) DOES NOT EXIST. SINCE THE GRAPH IS SIMPLE, EACH VERTEX CAN ONLY CONNECT TO 3 OTHER VERTICES AT MOST, SO DEG (EACH VERTEX) ≤ 3

(b) 

(c) DOES NOT EXIST $\underbrace{\text{DEG}(G)}_{\text{EVEN}} = 1+2+3+3 = 9 \leftarrow \text{ODD}$

(d) 

(e) DOES NOT EXIST. A TREE WITH 9 VERTICES MUST HAVE 8 EDGES (THEOREM 10.5.2)

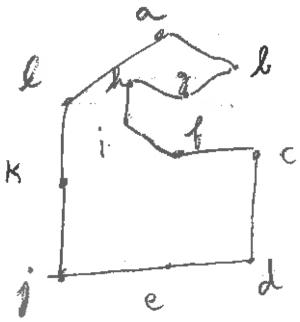
12. ~~13.~~ (a) Find an Euler Circuit in the graphs of exercises 16 and 17 on page 658, or show that such a circuit doesn't exist

(b) Find a Hamiltonian circuit in the graph of exercise 24 on page 659.

(a) ⑯ AN EULER CIRCUIT DOES NOT EXIST, BECAUSE NOTICE THAT ANY CIRCUIT THAT STARTS AT V_1, V_3, V_5 MUST SKIP V_0, V_2, V_4, AND ANY CIRCUIT THAT STARTS AT V_0, V_2, V_4 MUST SKIP V_1, V_3, V_5

⑰ DOES NOT EXIST BECAUSE DEG(C) IS ODD, BUT BY THEOREM 10.2.2, EVERY VERTEX MUST HAVE (POSITIVE) EVEN DEGREE.

(b) ONE POSSIBLE CIRCUIT IS a l k j e d c f i h g b a



13. (a) Is the following a tautology? Why or why not?

$$\Big[(p \vee q) \wedge ((\sim p) \vee r)\Big] \Rightarrow (q \vee r)$$

(b) Are the following two statements logically equivalent?

$$(\exists x P(x)) \wedge (\exists x Q(x)), \qquad \exists x (P(x) \wedge Q(x))$$

(a) TRUTH TABLE

p	q	r	$p \vee q$	$\sim p \vee r$	$(p \vee q) \wedge (\sim p \vee r)$	$q \vee r$	$\ldots \Rightarrow \ldots$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	F	T

SINCE $[(p \vee q) \wedge ((\sim p) \vee r)] \Rightarrow q \vee r$ IS ALWAYS TRUE FOR ALL TRUTH VALUES OF p, q, r, IT IS A TAUTOLOGY

(b) NO LET $P(x)$ BE THE STATEMENT $x^2 = 1$
AND $Q(x)$ BE THE STATEMENT $x^2 = 4$

THEN $\exists x \; P(x)$, NAMELY $x = 1$ (OR -1)
AND $\exists x \; Q(x)$, NAMELY $x = 2$ (OR -2)

BUT IT IS NOT TRUE THAT $\exists x \; (P(x) \wedge Q(x))$, THAT IS THERE IS NO x THAT SATISFIES BOTH $x^2 = 1$ AND $x^2 = 4$
(B/C OTHERWISE WE WOULD GET $1 = 4 \Rightarrow \Leftarrow$)

HENCE THE TWO STATEMENTS ARE NOT LOGICALLY EQUIVALENT.

14. Show that if a number of the form $2^n - 1$ is prime, then n must be prime as well.

Hint: Use the identity $a^M - b^M = (a-b)(a^{M-1} + ab^{M-2} + \cdots + a^{M-2}b + b^{M-2}$

~~$a^m - b^m = (a-b)(a^{m-1}b + a^{m-2}b^2 + \cdots + a^2b^{m-2} + ab^{m-1}$~~$)$

SUPPOSE $N = rs$ FOR SOME $r \geqslant 1$ AND $s \geqslant 1$ (INTEGERS)

WE NEED TO SHOW $r = 1$ OR $r = N$

BUT NOTICE THAT:

$$2^N - 1 = 2^{rs} - 1$$

$$= (2^r)^s - 1$$

USE THE HINT WITH $a = 2^r$, $b = 1$, $M = s$

$$= (2^r - 1)\left((2^r)^{s-1} + (2^r)^{s-2}1 + \cdots + (2^r)1^{s-2} + 1^{s-1}\right)$$

SO $2^r - 1$ IS A FACTOR OF $2^N - 1 \leftarrow \underline{\text{PRIME}}!$

AND HENCE $2^r - 1 = 2^N - 1 \implies 2^r = 2^N \implies \underline{r = N}$

<u>OR</u> $2^r - 1 = 1 \implies 2^r = 2^1 \implies \underline{r = 1}$

IN ANY CASE, $r = N$ OR $r = 1$, AND HENCE N IS PRIME.

1.5 [redacted] Show that for all positive integers m and n, we have

$$\lceil\frac{n}{m}\rceil = \lfloor\frac{n+m-1}{m}\rfloor$$

BY THE QUOTIENT-REMAINDER THEOREM WITH $d = M$,

WE GET $N = Mq + r$ WITH $0 \leq r < M$

1) FIRST LET'S DO THE CASE $\underline{r = 0}$

THEN $N = Mq$, SO $\left\lceil \frac{N}{M} \right\rceil = \left\lceil \frac{\cancel{M}q}{\cancel{M}} \right\rceil = \lceil q \rceil = q$

AND $\left\lfloor \frac{N+M-1}{M} \right\rfloor = \left\lfloor \frac{Mq+M-1}{M} \right\rfloor = \lfloor \overset{\text{INT!}}{\overbrace{q+1}} - \frac{1}{M} \rfloor$

FROM THE BOOK,

$\lfloor x + K \rfloor = \lfloor x \rfloor + K$ IF K IS AN INTEGER $\longrightarrow$ $= q + 1 + \lfloor -\frac{1}{M} \rfloor$

$= q + 1 - 1$ $\quad -1 \leq -\frac{1}{M} < 0$ (SINCE $M > 0$)

$= q$

SO IN THAT CASE, $\left\lceil \frac{N}{M} \right\rceil = \left\lfloor \frac{N+M-1}{M} \right\rfloor$ ✓

2) IN THE CASE $\underline{r > 0}$, WE GET

$\left\lceil \frac{N}{M} \right\rceil = \left\lceil \frac{Mq+r}{M} \right\rceil = \left\lceil \overset{\text{INT}}{q} + \frac{r}{M} \right\rceil = q + \lceil \underbrace{\tfrac{r}{M}}_{0 < \frac{r}{M} < 1} \rceil = \boxed{q + 1}$

AGAIN, IF K IS AN INTEGER, THEN $\lceil x + K \rceil = \lceil x \rceil + K$

EQUAL

AND $\left\lfloor \frac{N+M-1}{M} \right\rfloor = \left\lfloor \frac{Mq+r+M-1}{M} \right\rfloor = \left\lfloor \frac{Mq}{M} + \frac{r}{M} + 1 - \frac{1}{M} \right\rfloor$

$= \lfloor \underbrace{q+1}_{\text{INT}} + \frac{r-1}{M} \rfloor = q + 1 + \lfloor \underbrace{\tfrac{r-1}{M}}_{0 \leq \frac{r-1}{M} < 1 \text{ SINCE } r > 0} \rfloor = q + 1 + 0 = \boxed{q+1}$

16. (a) Show that for ~~all positive~~ NONNEGATIVE integers n, we have

$$\frac{1}{\sqrt{n+1}} \geq 2(\sqrt{n+2}-\sqrt{n+1})$$

Hint: Multiply both sides by $\sqrt{n+2}+\sqrt{n+1}$

(b) Prove by induction on n that for all positive integers n, we have

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} \geq 2(\sqrt{n+1}-1)$$

Cultural note: This inequality shows that the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges.

(a) WE NEED TO SHOW

$$\left(\frac{1}{\sqrt{N+1}}\right)\left(\sqrt{N+2}+\sqrt{N+1}\right) \geqslant 2\left(\sqrt{N+2}-\sqrt{N+1}\right)\left(\sqrt{N+2}+\sqrt{N+1}\right)$$

$\parallel$

$$2\left((\sqrt{N+2})^2-(\sqrt{N+1})^2\right)$$

$\parallel$

$$2\left(N+2-(N+1)\right)=2(N+2-N-1)$$

$\parallel$

$$2$$

$\parallel$

$$\frac{\sqrt{N+2}}{\sqrt{N+1}}+\frac{\sqrt{N+1}}{\sqrt{N+1}}$$

$\parallel$

$$\frac{\sqrt{N+2}}{\sqrt{N+1}}+1$$

SO WE NEED TO SHOW $\frac{\sqrt{N+2}}{\sqrt{N+1}}+1 \geqslant 2$, THAT IS $\frac{\sqrt{N+2}}{\sqrt{N+1}} \geqslant 1$

THAT IS $\sqrt{N+2} \geqslant \sqrt{N+1}$ THAT IS $N+2 \geqslant N+1$, THAT IS $2 \geqslant 1$

BUT IT IS ALWAYS TRUE THAT $2 \geqslant 1$, HENCE (a) IS ALWAYS TRUE

(b) SEE NEXT PAGE

PROBLEM 16(b) (CONTINUED)

LET P_N BE THE PROPOSITION

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{N}} \geq 2\left(\sqrt{N+1} - 1\right)$$

BASE CASE $N=1$

$$\frac{1}{\sqrt{1}} \overset{?}{\geq} 2\left(\sqrt{2} - 1\right) \; ?$$

$$1 \overset{?}{\geq} 2\left(\sqrt{2} - 1\right)$$

$$1 \overset{?}{\geq} 2\sqrt{2} - 2$$

$$3 \overset{?}{\geq} 2\sqrt{2}$$

$$9 \overset{?}{\geq} \left(2\sqrt{2}\right)^2 = 8 \; ? \quad \underline{\text{YES}}$$

HENCE P_1 IS TRUE

(ALTERNATIVELY, USE (a) WITH $N=0$)

INDUCTIVE STEP SUPPOSE P_N IS TRUE, THAT IS

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{N}} \geq 2\left(\sqrt{N+1} - 1\right)$$

SHOW P_{N+1} IS TRUE, THAT IS

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{N+1}} \geq 2\left(\sqrt{N+2} - 1\right)$$

BUT $\frac{1}{\sqrt{1}} + \quad + \frac{1}{\sqrt{N+1}}$

$$= \left(\frac{1}{\sqrt{1}} + \quad + \frac{1}{\sqrt{N}}\right) + \frac{1}{\sqrt{N+1}}$$

BY THE INDUCTIVE HYPOTHESIS

$$\geq 2\sqrt{N+1} - 1 + \frac{1}{\sqrt{N+1}}$$

BY (a)

$$\geq 2\left(\sqrt{N+1} - 1\right) + 2\left(\sqrt{N+2} - \sqrt{N+1}\right)$$

$$= 2\cancel{\sqrt{N+1}} - 2 + 2\sqrt{N+2} - 2\cancel{\sqrt{N+1}}$$

$$= 2\sqrt{N+2} - 2$$

$$= 2\left(\sqrt{N+2} - 1\right)$$

HENCE P_{N+1} IS TRUE, SO BY INDUCTION, P_N IS TRUE $\forall N$, THAT IS,

$$\frac{1}{\sqrt{1}} + \quad + \frac{1}{\sqrt{N}} \geq 2\left(\sqrt{N+1} - 1\right) \quad \forall N \geq 1$$

17. Use induction to prove that any integer $n \geq \overset{1}{\cancel{3}}$ can be written as $n = 2^k m$ for some odd integer m $\underset{\text{AND} > 0}{}$ and some non-negative integer k.

Hint: Argue in terms of whether n is even or n is odd.

WE WILL USE <u>STRONG</u> INDUCTION ON N.

LET P_N BE THE PROPOSITION "$N = 2^K M$ FOR SOME M ODD AND SOME $K \geq 0$"

<u>BASE CASE</u> <u>$N=1$</u> THEN $1 = 2^0(1)$, SO LET $K=0$ AND $M=1$ (ODD)

<u>IND STEP</u> SUPPOSE P_ℓ IS TRUE FOR ALL $\ell = 1, 2, \ldots, N-1$

THAT IS $\ell = 2^K M$ FOR SOME $K \geq 0$ AND $M > 0$ AND ODD

SHOW P_N IS TRUE, THAT IS $N = 2^K M$ FOR SOME $K \geq 0$ AND $\underset{\text{ODD}}{M > 0}$

<u>CASE 1</u> <u>N IS ODD</u>, BUT THEN $N = 2^0 N$, SO LET $K=0$ AND $M = N$ (ODD) SO P_N IS TRUE

<u>CASE 2</u> <u>N IS EVEN</u>, THEN $N = 2\ell$ FOR SOME $\ell = 1, 2, \ldots, N-1$

BUT THEN BECAUSE P_ℓ IS TRUE, $\ell = 2^{K'} M'$ FOR SOME $K' \geq 0$ AND $M' > 0$ ODD

BUT THEN $N = 2\ell = 2\, 2^{K'} M' = 2^{(K'+1)} M$

SO IF YOU LET $K = K'+1 \geq 0$, AND $M' = M$ (ODD), THEN WE GET

$N = 2^K M$, SO P_N IS TRUE IN THIS CASE TOO!

THEREFORE, IN BOTH CASES, P_N IS TRUE,

SO BY STRONG INDUCTION, P_N IS TRUE $\forall N$, THAT IS EVERY $N \geq 1$ CAN BE WRITTEN AS $N = 2^K M$ FOR SOME $M > 0$ ODD AND SOME $K \geq 0$

<u>NOTE</u> YOU COULD ALSO PROVE THIS USING THE FUNDAMENTAL THEOREM OF ARITHMETIC

NAMELY $N = 2^{\overset{K =}{\alpha_1}} \underbrace{p_2^{\alpha_2} \cdots p_{M'}^{\alpha_M}}_{M \text{ (ODD)}}$ FOR SOME PRIMES $p_2, \ldots, p_{m'} \neq 2$ ALL $\alpha_i \geq 0$

18) Consider the following Tower of Hanoi Problem: There are 3 poles in a row, and $\frac{n(n+1)}{2}$ disks, one disk of size 1, two of size 2, $\cdots$ and n of size n, where n is any positive integer. Initially one of the poles contains all the disks placed on top of each other in sets of decreasing size (so at the bottom there are the n plates of size n, then the $n-1$ plates of size $n-1$, until you get the 1 plate of size 1). Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be plaed on top of one of the same size. Let s_n be the minimum number of moves needed to transfer a tower of $\frac{n(n+1)}{2}$ disks from one pole to another.

Find a recurrence relation for s_n

SINCE EACH MOVE WAS NECESSARY

S_N = TOTAL NUMBER OF MOVES

$= S_{N-1} + N + S_{N-1}$

SO $\boxed{S_N = 2 S_{N-1} + N}$

TOP DISKS OF SIZE 1, 2, , N-1

N BOTTOM DISKS OF SIZE N

STEP 1 HAVE YOUR FRIEND TRANSFER THE TOP DISKS OF SIZE 1, 2, , N-1 TO B ; THIS TAKES S_{N-1} MOVES

STEP 2 TRANSFER THE BOTTOM N DISKS FROM A TO B

THIS TAKES N MOVES

STEP 3 FINALLY TRANSFER THE TOP DISKS FROM B TO C

THIS TAKES S_{N-1} MOVES

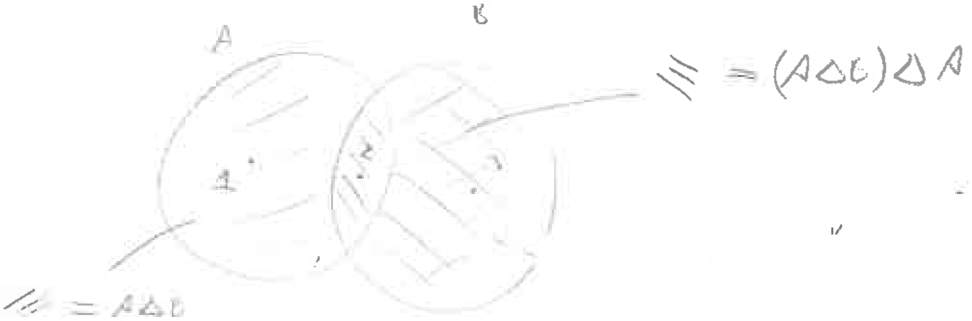
19. [redacted]. Define the symmetric difference of two sets by $A \Delta B = (A - B) \cup (B - A)$. Prove or disprove the following statements

[redacted]

(a) $(A \Delta B) \Delta A = A$

(b) $(A \Delta B)^c = A \cap B$

(a) FALSE



LET $A = \{1, 2\}$, $B = \{2, 3\}$

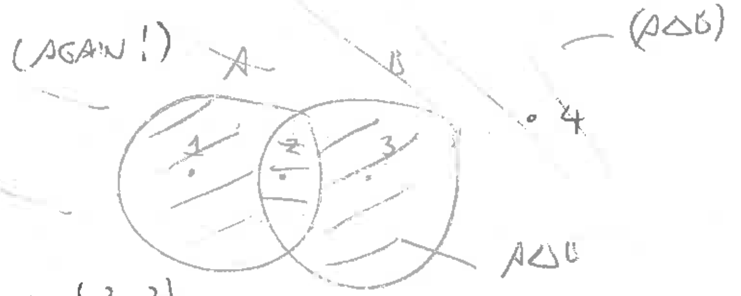
THEN $A \Delta B = (A - B) \cup (B - A) = \{1\} \cup \{3\} = \{1, 3\}$

$(A \Delta B) \Delta A = \{1, 3\} \Delta \{1, 2\} = (\{1, 3\} - \{1, 2\}) \cup (\{1, 2\} - \{1, 3\})$

$= \{3\} \cup \{2\} = \{2, 3\}$

$\{1, 3\} \neq \{2, 3\}$, SO $(A \Delta B) \Delta A \neq A$

(b) FALSE (AGAIN!)



$A = \{1, 2\}$, $B = \{2, 3\}$,

$A \Delta B = (A - B) \cup (B - A) = \{1\} \cup \{3\} = \{1, 3\}$

$(A \Delta B)^c = \{2, 4, \text{OTHER ELEMENTS}\}$, $A \cap B = \{2\}$

BUT $4 \in (A \Delta B)^c$ AND $4 \notin A \cap B$, SO $(A \Delta B)^c \neq A \cap B$

WITH $a \neq 0$

20) Show that the set of all real numbers x with the property that there exist integers a, b, c such that $ax^2 + bx + c = 0$ is countable.

OUR SET CAN BE WRITTEN AS

$$\bigcup_{(a,b,c) \in \mathbb{Z}^* \times \mathbb{Z} \times \mathbb{Z}} \{ x \in \mathbb{R} \mid ax^2 + bx + c = 0 \}$$

NOTICE THAT FOR GIVEN $(a,b,c) \in \mathbb{Z}^* \times \mathbb{Z} \times \mathbb{Z}$, THE SET $\{ x \in \mathbb{R} \mid ax^2 + bx + c = 0 \}$ IS COUNTABLE B/C THERE ARE AT MOST TWO SOLUTIONS TO $ax^2 + bx + c = 0$, NAMELY $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

MOREOVER, $\mathbb{Z}^* \times \mathbb{Z} \times \mathbb{Z}$ IS COUNTABLE BECAUSE THE PRODUCT OF COUNTABLE SETS IS COUNTABLE

THEREFORE OUR SET IS THE COUNTABLE UNION OF COUNTABLE SETS, AND THEREFORE IS COUNTABLE.

21. [redacted]. Define the following relation $\sim$ on $\mathbb{R}^2$:

$$(x,y) \sim (z,t) \iff x^2+y^2 = z^2+t^2$$

(a) Show that $\sim$ is an equivalence relation.

(b) What do the equivalence classes of $\sim$ look like?

(a) $(x,y) \sim (x,y)$ SINCE $x^2+y^2 = x^2+y^2$ ✓

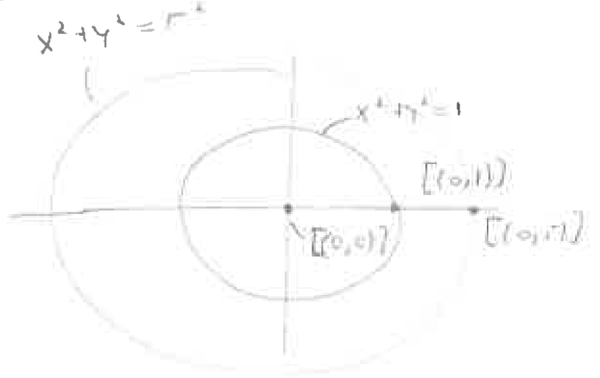
$(x,y) \sim (z,t) \Rightarrow (z,t) \sim (x,y)$ SINCE $x^2+y^2 = z^2+t^2$

$\Rightarrow z^2+t^2 = x^2+y^2$

$(x,y) \sim (z,t)$ & $(z,t) \sim (s,w) \Rightarrow (x,y) \sim (s,w)$

SINCE $x^2+y^2 = z^2+t^2$ & $z^2+t^2 = s^2+w^2 \Longrightarrow x^2+y^2 = s^2+w^2$

(b) THE EQUIVALENCE CLASSES ARE CIRCLES OF THE FORM $x^2+y^2=r^2$, WHERE $r \geq 0$ (IF $r=0$, THIS IS JUST $\{(0,0)\}$



$[(0,r)]$

(WHY? THE EQUIVALENCE CLASS OF $(0,r)$ IS $= \{(x,y) \in \mathbb{R}^2 \mid (x,y) \sim (0,r)\}$

$= \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 0^2+r^2\}$

$= \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = r^2\}$)

$$\textcircled{*} \quad \mathbb{R} = \underbrace{\bigcup_{\text{ALL} \neq \text{ EQUIV CLASSES } [x]}}_{\text{COUNTABLE (BY ASSUMPTION)}} \underbrace{[x]}_{\text{COUNTABLE (BY (b))}}$$

, SO $\mathbb{R}$ WOULD BE A COUNTABLE UNION OF COUNTABLE SETS, AND HENCE COUNTABLE $\Rightarrow\Leftarrow$

22. Define the relation $\sim$ on $A = \mathbb{R}$ by $x \sim y$ if and only if $x - y$ is rational.

(a) Show that $\sim$ is an equivalence relation

(b) Show that for every x, $[x]$ (the equivalence class of x) is countable
Hint: Show that $f : \mathbb{Q} \longrightarrow [x]$ defined by $f(q) = x + q$ is a bijection.

(c) Show that there are uncountably many *distinct* equivalence classes $[x]$, where $x \in \mathbb{R}$.

(a) $x \sim x$? IS $x - x$ RATIONAL? YES, BECAUSE $x - x = 0 \in \mathbb{Q}$ ✓

$x \sim y \Rightarrow y \sim x$? SUPPOSE $x - y = q$ WITH $q \in \mathbb{Q}$
THEN $y - x = -(x - y) = -q \in \mathbb{Q}$ ✓

$x \sim y$ & $y \sim z \Rightarrow x \sim z$? SPS $x - y = q$, AND $y - z = r$
WITH $q, r \in \mathbb{Q}$.

THEN $x - z = (x - y) + (y - z) = q + r \in \mathbb{Q}$ ✓

(b) 1) FIRST OF ALL, WE NEED TO MAKE SURE THAT f IS WELL-DEFINED (= "MAKES SENSE"), THAT IS, FOR EACH q, $f(q) \in [x]$
THAT IS, $f(q) - x$ IS RATIONAL.
BUT $f(q) - x = (x + q) - x = q \in \mathbb{Q}$, SO $f(q) - x$ IS RATIONAL ✓

2) f IS ONE-TO-ONE
SUPPOSE $f(q) = f(r)$, THEN $x + q = x + r$, SO $q = r$ ✓

3) f IS ONTO
SUPPOSE $y \in [x]$, THEN $y \sim x$ (BY DEF OF $[x]$), SO $y - x = q \in \mathbb{Q}$
FOR SOME q (BY DEF OF $\sim$), AND SO $y = x + q = f(q)$,
SO $y = f(q)$ FOR SOME $q \in \mathbb{Q}$. SINCE y WAS ARBITRARY, f IS ONTO.

(c) SUPPOSE THERE ARE ONLY COUNTABLY MANY DISTINCT EQUIVALENCE CLASSES
SINCE THE EQUIVALENCE CLASSES FORM A PARTITION OF $\mathbb{R}$, WE HAVE $\textcircled{*}$