MATH 453 – MIDTERM

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Name: ____

Start time (and date): ________________________

End time (and date): ________________________

Instructions: Welcome to your midterm! You have **180** (not 120) minutes to take this exam, for a total of 100 points. Write in full sentences whenever you can and try to be as precise as you can. If you need to continue your work on the back of a page, please indicate that you're doing so, or else your work may be discarded. Good luck, and may the maximum principle be with you! :)

Honor Code: I promise not to exceed the allotted time limit, not to communicate or collaborate with anyone during the exam-period, and I will not use any books or notes or cheat sheets or personal electronic devices (including calculators).

Signature: ____

1		25
2		25
3		25
4		25
Total		100

Date: Wednesday, March 15 – Friday, March 17, 2017.

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1. *(25 points)* State **and** prove the mean-value formula for Laplace's equation (both the sphere and the ball-version)

   Note: Remember that the volume of the ball $B(x, r)$ in $\mathbb{R}^n$ is $\alpha(n)r^n$ and that the surface area of the sphere $\partial B(x, r)$ in $\mathbb{R}^n$ is $n\alpha(n)r^{n-1}$

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2. *(25 points)*

(a) *(10 points)* State but do **not** prove the maximum principle for the heat equation (both statements)

Note: Remember that $U_T := U \times (0, T]$ (the parabolic cylinder) and $\Gamma_T := \overline{U}_T - U_T$ (the parabolic boundary)

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(b) *(15 points)* Use (a) to show that if U is connected, $T > 0$, and $g \in C(\Gamma_T)$ and $f \in C(U_T)$ are given, then there is at most one solution $u \in C^\infty(U_T) \cap C(\overline{U}_T)$ to the PDE

$$\begin{cases} u_t - \Delta u = f \text{ in } U_T \\ \qquad\quad u = g \text{ on } \Gamma_T \end{cases}$$

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3. *(25 points)* Show that if $f : \mathbb{R} \to \mathbb{R}$ is nonincreasing, then there is at most one solution $u = u(x) \in C(\overline{U})$ to the PDE

$$\begin{cases} -\Delta u = f(u) \text{ in } U \\ \qquad u = g \text{ on } \partial U \end{cases}$$

Note: There is no typo here, it is indeed $f(u)$ (f with input u), not $f(x)$.

Note: f is nonincreasing means that $(f(x) - f(y))(x - y) \leq 0$ for all x and y in $\mathbb{R}$.

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4. *(25 points)* Find an *explicit* solution (= not involving integrals) of the following heat equation for $n = 1$:

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u(x, 0) = e^{-x} \text{ in } \mathbb{R} \times \{t = 0\} \end{cases}$$

Hint: Write $u(x, t)$ in terms of integrals, expand the square, and pull out any terms that don't depend on y. Then complete the square of whatever you have remaining (with respect to y), pull out more terms that don't depend on y, and finally use a change of variable.

Note: Notice that once you have your solution, it should be very easy to check that it satisfies the PDE.

Note: Recall the fundamental solution of the heat equation is

$$\Phi(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

and also that $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$.

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