MATH 453 – MIDTERM – STUDY GUIDE

PEYAM RYAN TABRIZIAN

The Math 453 – Midterm will be given out on Wednesday, March 15, at 11:50 AM and due on Friday, March 17, at noon. It covers everything that we've learned, up to and including Theorem 5 (Uniqueness on bounded domains) of page 57 of the book. It will be a closed book and closed notes-exam, and no cheat-sheets will be allowed. You will have 2 hours (= 120 minutes) to take this exam. Also note that lecture on Friday, March 17 will be cancelled. I will have special office hours on Tuesday, March 14, from 4 to 5 and the TA will have a special TA Session on Tuesday, March 14, from 8 to 9 in 204 Clark (the TA session on Thursday, March 16 will be cancelled).

This is a study guide for the exam, and hopefully covers *everything* you need to know for the exam.

Note: You do **NOT** have to memorize the fundamental solutions for Laplace's equation and for the heat equation, I will provide them to you if necessary!

Main concepts: Laplace's equation (section 2.2) and Heat equation (section 2.3)

THEOREMS WHOSE STATEMENTS AND PROOFS YOU NEED TO KNOW (= MEMORIZE!!!)

Note: Don't worry too much about the smoothness assumptions (about continuity etc.), I'm more interested in the formulas/results. The way I stated and proved the results in lecture is completely fine, and you don't have to be as precise as the book. But of course, you have to say "Suppose $\Delta u = 0$ or suppose $u_t - \Delta u = 0$"

- Showing that Laplace's equation is invariant under rotations (Problem 2 in Chapter 2). I'd give you the definition of orthogonal matrix, just like on the HW.

Date: Wednesday, March 15, 2017.

- How to derive the fundamental solution of Laplace's equation (in the case $n \geq 3$) from the fact that $u(x) = v(r)$ for some radial v (pages 21-22)
- **The mean-value formulas for Laplace's equation** (both versions)
- The converse to the mean-value property (Theorem 3 on page 26)
- **The strong maximum principle for Laplace's equation**
- Using the strong maximum principle to show positivity (page 27)
- **Uniqueness of solutions of Poisson's equation using the maximum principle** (Theorem 5 on page 28)
- The decay estimates (Theorem 7 on page 29), but *only* in the case of $|u(x)|$ (see lecture)
- Liouville's theorem (Theorem 8 on page 30)
- Representation formula (Theorem 9 on page 30); you do **not** need to show that $\widetilde{u}(x)$ ($v(x)$ in lecture) is bounded.
- **Uniqueness of solutions of Poisson's equation using energy methods** (Theorem 16 on page 42)
- **Dirichlet's principle** (Theorem 17 on pages 42 and 43)
- (How to derive the fundamental solution of the heat equation) (with *plenty* of hints, though!)
- How to show that $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{\pi}$, and hence how to show that the integral of the fundamental solution is 1 (bottom of page 46).
- Infinite propagation speed of heat equation (page 48)
- Solution of homogeneous problem with general initial data (bottom of page 51)
- **The strong maximum principle for the Heat Equation** (Theorem 4 on page 55; I would provide you with the mean value formula for the heat equation if necessary)
- Infinite Propagation speed again (page 57)
- Uniqueness on bounded domains (Theorem 5 on page 57)

THEOREMS WHOSE <u>STATEMENTS</u> (BUT **NOT** PROOFS) YOU NEED TO KNOW (= MEMORIZE)

Note: Of course you have to know the statements of the theorems in the above sections as well!

- Divergence Theorem
- Dominated Convergence Theorem
- Polar Coordinates Formula
- Integration by parts (Theorems 1, 2, 3 on pages 711 and 712)
- Polar Coordinates formula (Theorem 4 on page 712)
- Solving Poisson's equation (Theorem 1 on page 23)

- Harnack's inequality (Theorem 11 on page 32)
- The decay estimates (Theorem 7 on page 29), but only in the case of the first derivative (see lecture)
- Smoothness (Theorem 6 on page 28)
- The true second-derivative test (see handout)
- Properties of mollifiers (Theorem 7(i) and 7(ii) on page 714, don't worry about what U_ϵ is)
- Cauchy-Schwarz and Cauchy's inequalities ((a) and (i) on pages 706 and 708)
- Solution of the initial-value problem of the heat equation (Theorem 1 on page 47; again, don't worry too much about the assumptions)
- Solution of the nonhomogeneous problem of the heat equation (Theorem 2 on page 50)

CHAPTER 1: INTRODUCTION

- This chapter is just an overview of PDEs, so don't worry about it
- Know what $D^\alpha u$ means, where α is a multi-index
- Do **not** worry about Problem 5 in Chapter 1 (which was on HW 1), I will not ask you about it

CHAPTER 2: LAPLACE'S EQUATION

Introduction.

- Don't worry about the physical derivation of Laplace's equation (in terms of net flux)
- Know the fact that if $\int_V f dx = 0$ for all regions V, then $f \equiv 0$
- Know the divergence theorem

Section 2.2.1: Fundamental Solution.

- Know how to show that Laplace's equation is invariant under rotation (Problem 2 in Chapter 2)
- Know how to derive the fundamental solution of Laplace's equation (in the case $n \geq 3$) from the fact that $u(x) = v(r)$ (pages 21-22)
- Know the definition of convolution.
- Know the dominated convergence theorem, the polar coordinates formula, and the integration by parts formula, and how to apply them. The examples covered in Lecture 4, on HW 2, and Problems 1 and 2 in HW3 give you great practice with those concepts.
- Know the definition of the normal derivative, and in particular the formula for ν on $\partial B(0,r)$ and $\partial B(x,r)$.
- Know the formula for the solution of Poisson's equation, but you do **not** need to memorize the lengthy proof I gave you. That said, do

at least understand the techniques used; I might in theory ask you a question that uses the techniques (like integration by parts)

- Know how to show that if f is continuous, then the average integral of f on $\partial B(x, r)$ goes to $f(x)$ (see HW 3)

Section 2.2.2: Mean-Value Formulas.

- Be comfortable with change of variables! Notice how much we've used this throughout the course!
- Know the statement and the proof of the mean-value formulas for Laplace's equation
- Don't worry about Problem 3 in Chapter 2 (the mean-value formula with f and g) it's awkward to ask that on an exam

Section 2.2.3: Properties of harmonic functions.

- Know the statement and the proof of the strong (and weak) maximum principle. Also know how to prove it the way they do in Problem 4 of Chapter 2.
- Know how to apply the strong maximum principle to show that a solution u (with $g \geq 0$) is positive (see page 27)
- Know how to apply the strong maximum principle to show that solutions of Poisson's equation on a bounded open connected domain are unique (Theorem 5 on page 28)
- Know the statement of Harnack's inequality (Theorem 11 on page 32). Don't memorize the proof, but notice how the mean-value property is used here!
- For the decay estimates, only know the statement and proof for $|u(x)|$ itself (just like I did in lecture). For the first derivative $|Du(x)|$, know the statement, but not the proof.
- Know Liouville's theorem (Theorem 8 on page 30) and the Representation formula (Theorem 9 on page 30); in that proof you do **not** need to show that v ($\widetilde{u}(x)$ in the book) is bounded.
- Know the statement about smoothness (Theorem 6 on page 28), but do **not** know the proof. You do **not** need to know the definition of $\eta(x)$ or $\eta_\epsilon(x)$, but you **do** need to know what f_ϵ is and the two properties it satisfies (that it's smooth and that it converges to f)
- Know how to show that a function is subharmonic (See Problems 5(c)(d) in Chapter 2). Also know how to do Problem 6 in Chapter 2 (I'd provide you with the hint they give in the book).

Section 2.2.5: Energy methods.

- Know how to show how to prove uniqueness of Poisson's equation using energy methods.

- Problem 3 on HW 2 provides more practice with energy methods!
- Know the Cauchy-Schwarz inequality and Cauchy's inequalities
- Know how to derive Dirichlet's principle; I may also ask you about more exotic cases like the additional problem on HW 5.
- Know how to show that if $\int_W fg = 0$ for all smooth g, then $f \equiv 0$.

THE HEAT EQUATION

Introduction.

- Again, don't worry about the physical derivation of the heat equation

Section 2.3.1: Fundamental Solution.

- You don't have to worry about how to guess that $u(x,t) = \frac{1}{t^{\frac{n}{2}}} v\left(\frac{x}{\sqrt{t}}\right)$, but I *could* ask you how to derive the fundamental solution *if* you have this form. That said, I'd give you plenty of hints, like what Δw is, or that you have to multiply by r^{n-1}.
- Know the formula, but not the proof, for the solution of the initial-value problem of the heat equation (Theorem 1 on page 47); *especially* don't worry about the geometric lemma. But again, some of the techniques in the proof are useful, like writing $g(x_0)$ in terms of an integral and estimating that integral, and change of variables
- Know how to show that the solution of the heat equation has infinite propagation speed.
- Know the formula, but not the proof, of the solution of the nonhomogeneous problem. That said, again, some of the techniques are very classical (such as integration by parts).
- Dont' worry about the chain rule for integrals
- That said, know how to derive the solution of the general inhomogeneous problem (bottom of page 51). This technique used there (where you subtract two solutions) is very important
- Don't worry about Problem 13 in Chapter 2
- I could ask you to do something like Problem 14 in Chapter 2, where you transform an equation into the heat equation.

Section 2.3.2: Mean-value formula.

- **I will provide you with the definitions of parabolic cylinder and parabolic boundary, so don't worry about them.**
- Don't worry about the definition of the heat ball, I will provide it to you if necessary
- Look at the statement of the mean-value formula for the heat equation (Theorem 3 on page 53), but *don't* memorize it. I'd provide

you with the statement if necessary. You can also skip the proof, because it is a bit ridiculous!

- **Know the statement and the proof of the strong maximum principle for the heat equation**, but don't worry too much about the second part of the proof about line segments; it would be totally ok to say "Cover U_{t_0} with heat balls," like the way I did in lecture.
- Know how to derive the infinite propagation speed of the heat equation using the maximum principle (top of page 57)
- Know how to derive uniqueness of the heat equation on bounded domains (Theorem 5 on page 57); I would like to have a little bit more details than in the book, just the way I did in lecture

Note: I may not get to the last two points in lecture, but you are still responsible for them, just because they are so similar to the case for Laplace's equation!