SOLUTIONS

MATH 453 – MIDTERM

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Name: ______________________________

Start time: ____________________

End time: ____________________

Instructions: Welcome to your midterm! You have 180 (not 120) minutes to take this exam, for a total of 100 points. Write in full sentences whenever you can and try to be as precise as you can. If you need to continue your work on the back of a page, please indicate that you're doing so, or else your work may be discarded. Good luck, and may the maximum principle be with you! :)

Honor Code: I promise not to exceed the time limit, not to communicate or collaborate with anyone during the exam-period, and I will not use any books or notes or cheat sheets or personal electronic devices (including calculators).

Signature: ______________________________

1		25
2		25
3		25
4		25
Total		100

Date: Wednesday, March 15 – Friday, March 17, 2017.

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1. *(25 points)* State and prove the mean-value formula for Laplace's equation (the sphere and the ball-version)

Note: Remember that the volume of the ball $B(0,r)$ in $\mathbb{R}^n$ is $\alpha(n)r^n$ and that the surface area of the sphere $\partial B(0,r)$ is $n\alpha(n)r^{n-1}$

MEAN-VALUE FORMULAS IF $u \in C^2(U)$ IS HARMONIC, THEN

1) $$u(x) = \fint_{\partial B(x,r)} u(y)\, ds(y)$$

2) $$u(x) = \fint_{B(x,r)} u(y)\, dy$$

FOR ALL BALLS $B(x,r) \subseteq U$

PROOF OF 1) LET $$\varphi(r) = \fint_{\partial B(x,r)} u(y)\, ds(y)$$

$$= \frac{1}{N\alpha(N) r^{N-1}} \int_{\partial B(x,r)} u(y)\, ds(y)$$

$z = \frac{y-x}{r}$

$ds(z) = \frac{1}{r^{N-1}} ds(y)$

$$= \frac{1}{N\alpha(N)\cancel{r^{N-1}}} \cancel{r^{N-1}} \int_{\partial B(0,1)} u(x+rz)\, ds(z)$$

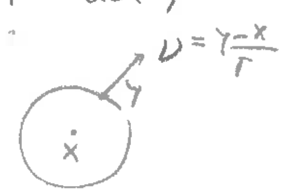
$$= \frac{1}{N\alpha(N)} \int_{\partial B(0,1)} u(x+rz)\, ds(z)$$

HENCE $$\varphi'(r) = \frac{1}{N\alpha(N)} \int_{\partial B(0,1)} Du(x+rz) \cdot z\, ds(z)$$

$y = x + rz$

$ds(y) = r^{N-1} ds(z)$

$$= \frac{1}{N\alpha(N) r^{N-1}} \int_{\partial B(x,r)} Du(y) \cdot \underbrace{\left(\frac{y-x}{r}\right)}_{\nu} ds(z)$$



$$= \frac{1}{N\alpha(N) r^{N-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\, ds(z)$$

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$$= \frac{1}{N\alpha(N) r^{N-1}} \int_{B(x,r)} \underbrace{\Delta u}_{=0} \, dy$$

$$= 0$$

HENCE $\varphi'(r) = 0$, SO φ IS CONSTANT, AND SO

$$\varphi(r) = \fint_{\partial B(x,r)} u(y)\, ds(y) = \lim_{t \to 0} \varphi(t) = \lim_{t \to 0} \fint_{\partial B(x,t)} u(y)\, ds(y) = u(x)$$

$$\Rightarrow \fint_{\partial B(x,r)} u(y)\, ds(y) = u(x)$$

PROOF OF 2) USING POLAR COORDINATES,

$$\fint_{B(x,r)} u(y)\, dy = \frac{1}{\alpha(N) r^N} \int_{B(x,r)} u(y)\, dy$$

$$= \frac{1}{\alpha(N) r^N} \int_0^r \left(\int_{\partial B(x,s)} u(y)\, ds(y) \right) ds$$

$$= \frac{1}{\cancel{\alpha(N)} r^N} \int_0^r u(x)\, N \cancel{\alpha(N)}\, s^{N-1}\, ds$$

$$= \frac{\cancel{N}\, u(x)}{\cancel{r^N}} \left(\int_0^r s^{N-1}\, ds \right) = \frac{\cancel{r^N}}{\cancel{N}}$$

$$= u(x)$$

2. *(25 points)*

   (a) *(10 points)* State but do **not** prove the maximum principle for the heat equation (both statements)

   Note: I'd like to remind you that $U_T := U \times (0, T]$ (the parabolic cylinder) and $\Gamma_T := \overline{U}_T - U_T$ (the parabolic boundary)

IF $U \in C^{\alpha}(U_T) \cap C(\overline{U_T})$ SOLVES $U_t - \Delta U = 0$ IN U_T, THEN

(1) $\max_{\overline{U_T}} U = \max_{\Gamma_T} U$

(2) IF U IS CONNECTED AND $\exists\, (x_0, t_0) \in U_T$ SUCH THAT

$U(x_0, t_0) = \max_{\overline{U_T}} U$

THEN U IS CONSTANT IN $\overline{U}_{t_0}$

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(b) ([redacted] *points)* Use (a) to show that there is at most one solution to the equation

$$\begin{cases} u_t - \Delta u = f \text{ in } U_T \\ u = g \text{ on } \Gamma_T \end{cases}$$

where U is connected, $g \in C(\Gamma_T)$, $f \in C(U_T)$ and $u \in C^\infty(U_T) \cap C(\overline{U}_T)$.

SUPPOSE U SOLVES $\begin{cases} U_t - \Delta U = f \\ U = g \end{cases}$

AND V SOLVES $\begin{cases} V_t - \Delta V = f \\ V = g \end{cases}$

LET $W = U - V$

THEN $W_t - \Delta W = (U_t - \Delta U) - (V_t - \Delta V) = f - f = 0$ IN U_T

$W = g - g = 0$ ON Γ_T

SO W SOLVES $\begin{cases} W_t - \Delta W = 0 & \text{IN } U_T \\ W = 0 & \text{ON } \Gamma_T \end{cases}$

BY THE WEAK MAXIMUM PRINCIPLE

$\max_{\overline{U_T}} W = \max_{\Gamma_T} \underbrace{W}_{0} = 0$ SO $W \leq 0$ IN $\overline{U_T}$

SIMILARLY $\min_{\overline{U_T}} W = \min_{\Gamma_T} W = 0$ SO $W \geq 0$ IN $\overline{U_T}$

HENCE $W \equiv 0$ IN $\overline{U_T}$, SO $U \equiv V$ IN $\overline{U_T}$.

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3. *(25 points)* Show that if $f : \mathbb{R} \to \mathbb{R}$ is nonincreasing, then there is at most one solution $u = u(x) \in C(\overline{U})$ to the PDE

$$\begin{cases} -\Delta u = f(u) \text{ in } U \\ \quad u = g(x) \text{ on } \partial U \end{cases}$$

Note: This is not a typo; it's indeed $f(u)$, not $f(x)$.

Note: f is nonincreasing if and only if $(f(x) - f(y))(x - y) \leq 0$ for any x and y in $\mathbb{R}$.

SUPPOSE U SOLVES $\begin{cases} -\Delta U = f(U) \\ U = g \end{cases}$

AND V SOLVES $\begin{cases} -\Delta V = f(V) \\ V = g \end{cases}$

SUBTRACT BOTH EQUATIONS TO GET

$$-\Delta(U-V) = f(U) - f(V)$$

MULTIPLY THIS BY $U-V$ AND INTEGRATE OVER U

$$\int_U -\Delta(U-V)\,(U-V) = \int_U \underbrace{(f(U)-f(V))(U-V)}_{\leq 0 \text{ (BY ASSUMPTION)}}$$

IBP

$$\int_U D(U-V)\cdot D(U-V) \leq 0$$

(NO BOUNDARY TERMS BECAUSE $U-V = g-g = 0$ ON ∂U)

$\int_U |D(U-V)|^2 \leq 0$, BUT SINCE $\int_U \underbrace{|D(U-V)|^2}_{\geq 0} \geq 0$,

WE GET $\int_U |D(U-V)|^2 = 0$ AND HENCE $D(U-V) \equiv 0$

HENCE $U-V \equiv C$ BUT SINCE $U-V \equiv 0$ ON ∂U, WE GET $C=0$

HENCE $U \equiv V$ IN $\overline{U}$.

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4. *(25 points)* Find an *explicit* solution (= not involving integrals) of the following heat equation for $n = 1$:

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u(x,0) = e^{-x} \text{ in } \mathbb{R} \times \{t = 0\} \end{cases}$$

Hint: At some point, it's useful to complete the square with respect to y and use a change of variable.

Note: Recall the the fundamental solution of the heat equation is

$$\Phi(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

and also that $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$.

BY THE FORMULA FOR THE SOLUTION TO THE INITIAL-VALUE PROBLEM:

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t) e^{-y} \, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-y)^2}{4t}} e^{-y} \, dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2 - 4ty}{4t}} \, dy$$

BUT $$\frac{-(x-y)^2 - 4ty}{4t} = \frac{-x^2 + 2xy - y^2 - 4ty}{4t} = \frac{-x^2}{4t} - \frac{(y^2 - 2xy + 4ty)}{4t}$$

$$= \frac{-x^2}{4t} - \frac{(y - (x-2t))^2}{4t} + \frac{(x-2t)^2}{4t}$$

HENCE $$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2 + (x-2t)^2}{4t}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x-2t))^2}{4t}} \, dy$$

$z = y - (x-2t)$

$$= \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2 + x^2 - 4xt + 4t^2}{4t}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}} \, dy$$

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$$= \frac{1}{\sqrt{4\pi \cancel{t}}} e^{\frac{-4xt+4t^2}{4t}} \cancel{\sqrt{4t}} \int_{-\infty}^{\infty} e^{-(z')^2} dz' \qquad z' = \frac{z}{\sqrt{4t}}$$

$$= \frac{1}{\cancel{\sqrt{\pi}}} e^{-x+t} \cancel{\sqrt{\pi}}$$

$$= e^{t-x}$$

$$\boxed{u(x,t) = e^{t-x}}$$