SOLUTIONS

MATH 200 – MIDTERM 1

PEYAM RYAN TABRIZIAN

Name: ______________________________

Instructions: Welcome to Midterm 1! You have 50 minutes to take this exam, for a total of 100 points. **Do not open the exam until instructed to do so**. Remember that you are not only graded on the correctness of your answer, but also on the clarity and completeness of your proofs. Write in complete sentences whenever you can. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. May your luck be prime! :)

Honor Code: I promise not to communicate or collaborate with anyone during the exam, and I will not use any books or notes or cheat sheets or personal electronic devices (**including calculators**).

Signature: ______________________________

1		20
2		25
3		30
4		25
Total		100

Date: Friday, March 10, 2017.

1. *(20 points)* Fill out the following truth table (feel free to add more columns if that helps you)

p	q	r	$p \Rightarrow (q \wedge r)$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

2. *(25 points)* Show that if p is prime, then $\sqrt{p}$ is irrational.

Note: You are allowed to use (without proof) that if p is prime and p divides n^2, then p divides n (where n is an integer)

SUPPOSE $\sqrt{p}$ IS RATIONAL, THAT IS $\sqrt{p} = \frac{a}{b}$

FOR $a, b \in \mathbb{Z}$, $b \neq 0$

WLOG, ASSUME a & b HAVE NO FACTORS IN COMMON

THEN $\sqrt{p} = \frac{a}{b} \Rightarrow a = \sqrt{p}\, b \Rightarrow a^2 = p\underbrace{b^2}_{=K \in \mathbb{Z}} = Kp$

HENCE p DIVIDES a^2 HENCE p DIVIDES a, SO $a = Lp$

FOR SOME $L \in \mathbb{Z}$

BUT THEN $a^2 = pb^2 \Rightarrow (Lp)^2 = pb^2$

$\Rightarrow L^2 p^2 = pb^2$

$\Rightarrow b^2 = p\underbrace{L^2}_{=K' \in \mathbb{Z}} = K'p$

HENCE p DIVIDES b^2, SO p DIVIDES b, SO $b = L'p$ FOR SOME $L' \in \mathbb{Z}$

BUT THEN a & b HAVE A FACTOR OF p IN COMMON! $\Rightarrow\Leftarrow$

HENCE $\sqrt{p}$ IS IRRATIONAL

3. *(30 points)* Show that, for all integers $n \geq 1$, we have:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

Hint: At some point, you'll need to use $n(n+1) = n^2 + n \geq n^2$.

Note: In this problem, you are not only graded on the correct answer, but also on the way your write down your answer, so make sure to include all your steps.

INDUCTION

LET P_N BE THE PROPOSITION

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{N}} \geq \sqrt{N}$$

BASE CASE $N=1$ THEN $\frac{1}{\sqrt{1}} = 1 \geq \sqrt{1}$ ✓

INDUCTIVE STEP SUPPOSE P_N IS TRUE, THAT IS

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{N}} \geq \sqrt{N}$$

SHOW P_{N+1} IS TRUE, THAT IS $\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{N+1}} \geq \sqrt{N+1}$

BUT $\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{N+1}}$

$$= \left(\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{N}}\right) + \frac{1}{\sqrt{N+1}}$$

$$\geq \sqrt{N} + \frac{1}{\sqrt{N+1}} \quad \text{(B/C } P_N \text{ IS TRUE)}$$

$$= \frac{\sqrt{N}\sqrt{N+1} + 1}{\sqrt{N+1}} \qquad N+1 = \left(\sqrt{N+1}\right)^2$$

BY HINT

$$= \frac{\sqrt{N(N+1)} + 1}{\sqrt{N+1}} \geq \frac{\sqrt{N^2} + 1}{\sqrt{N+1}} = \frac{N+1}{\sqrt{N+1}} = \sqrt{N+1} \checkmark$$

HENCE P_{N+1} IS TRUE

HENCE P_N IS TRUE $\forall N \geq 1$.

4. *(25 points)* Show that if x is a real number and m is a positive integer such that $x - \lfloor x \rfloor \geq \frac{m-1}{m}$, then

$$\lfloor mx \rfloor = m \lfloor x \rfloor + (m-1)$$

Note: The quotient-remainder theorem does not help, so don't even go there...

LET $N = \lfloor X \rfloor$, THEN $\underline{N \leq x < N+1}$ $(*)$

WE WANT TO SHOW $\lfloor Mx \rfloor = M\lfloor x \rfloor + (M-1) = MN + (M-1)$

THAT IS $MN + (M-1) \leq Mx < MN + M$

MULTIPLYING $(*)$ BY M, WE GET $\underline{MN \leq Mx < M(N+1)} = MN + M$ (1)

ON THE OTHER HAND $X - \lfloor X \rfloor \geq \frac{M-1}{M}$

$\Rightarrow X - N \geq \frac{M-1}{M}$

$\Rightarrow Mx - MN \geq M-1$

$\Rightarrow \underline{Mx \geq MN + (M-1)}$ (2)

COMBINING (1) & (2) WE GET $MN + (M-1) \leq Mx < MN + M$

WHICH IS WHAT WE WANTED TO SHOW.