Math 200 – Lecture 5 – Multiple Quantifiers

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1 Multiple Quantifiers

On our previous exciting episode, we discussed what it means for a statement to have a quantifier, and we introducted the two most important ones: $\forall$ (for all) and $\exists$ (there exists), and we also discussed how to negate them. Let's start with an example to remind ourselves what we did:

Example: The following statement is true

$$\exists n \in \mathbb{N} \text{ such that } n^2 = 2n - 1$$

Namely $n = 1$.

Let's try to negate it:

$$\text{Not } \left(\exists n \in \mathbb{N} \text{ such that } n^2 = 2n - 1\right)$$

By the rules discussed last time, this is equivalent to:

$$\forall n \in \mathbb{N} \text{ Not } \left(n^2 = 2n - 1\right)$$

Which is the same as

$$\forall n \in \mathbb{N} \; n^2 \neq 2n - 1$$

Now of course, just as in life, things in math are not as simple as they seem, and a lot of statements in math depend on more than one variable, such as:

Example: The statement $y^2 = x$ depends on x **and** y.

The amazing thing is that you can use quantifiers even in the case of several variables (and this is called multiple quantification). There are four possible scenarios here:

Definition 1: $\forall x \forall y$ Blah means "For all x and y, Blah" (sometimes written as $\forall x, y$ Blah)

Example: The following is a true statement: $\forall x, y \in \mathbb{R}, x^2 + y^2 \geq 0$

Definition 2: $\forall x \exists y$ Blah means "For all x there is a y such that Blah"

Important: Here y (usually) *depends* on x !

Example: $\forall x \geq 0 \, \exists y \geq 0$ such that $y^2 = x$, namely $y = \sqrt{x}$. Notice that here y depends on x.

(By the way, notice how this combines the rules we talked about on Friday: $\forall x$ means that you let x be arbitrary, whereas $\exists y$ means you need to find an explicit y, so here you let $y = \sqrt{x}$, where x is arbitrary; so somehow here you find a y but you don't touch x)

Definition 3: $\exists x \forall y$ Blah means that there is a **special** x such that for all y Blah

IMPORTANT: x does **NOT** depend on y!!!!

Example: $\exists x > 0 \, \forall y > 0, xy \geq y$, **namely** $x = 1$ (because then, **no matter what** y is, we have $xy \geq y$).

Definition 4: $\exists x \exists y$ Blah means "There exists x and y such that Blah" (sometimes written as $\exists x, y$ Blah)

Example: $\exists x, y \in \mathbb{R}$ such that $(x + y)^2 = x^2 + y^2$, namely $x = 1$ and $y = 0$.

Remark 1: $\forall x \exists y$ Blah is **NOT** the same as $\exists x \forall y$ Blah. In the first case, the y (usually) depends on x, whereas in the second case, the y does **not** depend on x.

Example: "$\forall$ Student $\exists$ HW such that Student gets an A on HW" means that every student is lucky enough to have a HW assignment on which he/she gets an A. For example, Student 1 could get an A on HW 3, Student 2 on HW 5, etc. Here HW depends on Student, might be different for each student.

BUT "$\exists$ Student such that $\forall$ HW, Student gets an A on HW" means that there is some smart student who gets an A on every single homework assignmnent. Here HW does **not** depend on Stud.

Remark 2: $\forall x \exists y$ Blah is **NOT** the same as $\exists y \forall x$ Blah (don't interchange those!)

Example: $\forall$ Student $\exists$ HW such that Student gets an A on HW is not the same as $\exists$ HW such that $\forall$ Student, Student gets an A on HW. The latter means that there is some easy assignment HW* (say HW 5) on which everyone gets an A.

Remark 3: BUT it *is* true that

$$\exists y \forall x \text{ Blah} \Rightarrow \forall x \exists y \text{ Blah}$$

Example: It is true that if $\exists$ HW such that $\forall$ Student, Student gets an A on HW, then $\forall$ Student $\exists$ HW such that Student gets an A. Namely in the latter case, you let $HW = HW^\star$ (the easy assignment above). Then for all students, there is a HW, namely the easy assignment, where Student gets an a on HW. But the first statement is more specific, it requires the HW to be independent of the student.

Remark 4: $\forall x \forall y$ Blah is the same as $\forall y \forall x$ Blah, and $\exists x \exists y$ Blah is the same as $\exists y \exists x$ Blah.

What the second statement means is that there is some student and some homework such that the student gets an A on the HW.

And the thing is that quantifiers can get quite complicated!

Example: (not on the exam, just for illustrative purposes): What makes analysis so complicated is that you have quantifiers inside quantifiers. For instance, the rigorous definition of

$$\lim_{x \to 2} f(x) = 3$$

is:

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, |x-2| < \delta \text{ and } x \neq 2 \Rightarrow |f(x) - 3| < \epsilon$$

Here δ depends on ϵ (because the $\exists \delta$ is after the $\forall \epsilon$), but δ does not depend on x (because the $\forall x$ is after the $\exists \delta$).

So usually what you say in Analysis is: Let $\epsilon > 0$ be arbitrary, let $\delta = \sqrt{\epsilon}$ (depends on ϵ but not on x), and let x be arbitrary $\cdots$

2 Negating Quantifiers

Straightforward *if* you remember the rules from last time. Remember that to negate "for all" you replace it by "there exists" and you put the "not" inside. It's the same thing here:

Example: Not($\forall x \exists y, y = x + 1$) is the same as $\exists x \forall y, y \neq x + 1$.

Example: Not ($\forall$ HW $\exists$ Student such that Student gets an A on HW) is the same as $\exists$ HW such that $\forall$ Students, Student does not get an A on the HW, namely there is a very brutal homework where no one gets an A.

Example: Not ($\forall x \exists y \exists z \forall t \exists w, x + y + z = t + w$) is the same as

$$\exists x \forall y \forall z \exists t \forall w, x + y + z \neq t + w$$

Example: Is it true that $\forall a \in \mathbb{R}\ \exists x \in \mathbb{R}$ such that $\sin(x) = a$? (by the way, don't freak out, the trig in this example is just for illustrative purposes, there won't be any trig required in this class)

I'm claiming that the answer is no; to do this, you have to show Not ($\forall a \in \mathbb{R} \exists x \in \mathbb{R}, \sin(x) = a$) is true, that is

$$\exists a \in \mathbb{R}\ \forall x \in \mathbb{R}, \sin(x) \neq a.$$

Namely let $a = 2$ (notice this does not depend on x), then $\sin(x) = 2$ has no solutions, so $\sin(x) \neq 2$.

3 Arguments with quantified statements

This is basically section 3.4, the only point of this section being that basically all the things we learned about proof techniques (like modus ponens etc.) carry out to quantified statements. The book again gives a long list of formulas, but you only need to know the ones here:

1) Universal Instantiation: $\forall x P(x) \Rightarrow P(a)$

Example:

$$\forall x \in \mathbb{R}, x^2 \geq 0 \Rightarrow 3^2 \geq 0$$

Example: Every student gets an A $\Rightarrow$ You get an A

2) Universal Modus Ponens: ($\forall x P(x) \Rightarrow Q(x)$ and $P(a)$) implies $Q(a)$

Example: All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

3) Universal Modus Tollens: ($\forall x P(x) \Rightarrow Q(x)$ and $\sim Q(a)$) implies $\sim P(a)$.

Example: All men are mortal. Zeus is not mortal. Therefore, Zeus is not a man.

Last but not least, just as before, there are converse and inverse errors.

Converse Error: $(\forall x P(x) \Rightarrow Q(x))$ and $Q(a)$ does not imply $P(a)$.

Example: The following statement is incorrect: All thieves were at Purple Pub. John was at Purple Pub. Therefore John is a thief. (Sorry John! It is incorrect because there are people at Purple Pub who are not thieves).

Inverse Error: $(\forall x P(x) \Rightarrow Q(x))$ and $\sim P(a)$ does not imply $\sim Q(a)$.

Example: The following statement is incorrect. All thieves were at Purple Pub. John is not a thief. Therefore, John is not at Purple Pub (It is incorrect because again there are non-thieves at Purple Pub).

But it is true that if John is not at Purple Pub, then John is not a thief (that is just modus tollens).

Conclusion: Congratulations, we are officially done with Chapter 3. That also means that we'll start Chapter 4, Introduction to Proofs. Expect this course to get a bit harder (as compared to the previous 3 chapters).