HW 9 – SOLUTIONS

#24 (a) $(k(t) + p(t))'$

$$= k'(t) + p'(t)$$

$$= \frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} U_x^2(x,t)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} U_t^2(x,t)\,dx\right)$$

$$= \frac{1}{\cancel{2}}\int_{-\infty}^{\infty} \cancel{2}U_x U_{xt} + \cancel{2}U_t U_{tt}\,dx$$

$$= \int_{-\infty}^{\infty} U_x U_{xt} + U_t U_{xx}\,dx$$

IBP

$$= \int_{-\infty}^{\infty} \cancel{U_x U_{xt}} - \cancel{U_{tx} U_x}\,dx$$

$$= 0$$

HENCE $(k(t) + p(t))' = 0$, so $k(t) + p(t)$ is CONSTANT

(b) BY D'ALEMBERT,

$$U(x,t) = \frac{1}{2}\left(g(x-t) + g(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$$

HENCE $U_x(x,t) = \frac{1}{2}\left(g'(x-t) + g'(x+t)\right) + \frac{1}{2}\left(h(x+t) - h(x-t)\right)$

AND $U_t(x,t) = \frac{1}{2}\left(-g'(x-t) + g'(x+t)\right) + \frac{1}{2}\left(h(x+t) + h(x-t)\right)$

THEN $\quad k(t) - p(t) = \frac{1}{2}\int_{-\infty}^{\infty} (U_x)^2 - (U_t)^2$

$$= \frac{1}{2}\int_{-\infty}^{\infty} (U_x - U_t)(U_x + U_t)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{2}\Big(g'(x-t) + \cancel{g'(x+t)} + g'(x-t) - \cancel{g'(x+t)} + \cancel{h(x+t)} - h(x-t) - \cancel{h(x+t)} - h(x-t)\Big)$$

$$\times\ \frac{1}{2}\Big(\cancel{g'(x-t)} + g'(x+t) - \cancel{g'(x-t)} + g'(x+t) + h(x+t) - \cancel{h(x-t)} + h(x+t) + \cancel{h(x-t)}\Big)$$

$$k(t) - p(t) = \frac{1}{2}\int_{-\infty}^{\infty} \big(g'(x-t) - h(x-t)\big)\big(g'(x+t) + h(x+t)\big)\,dx$$

NOW LET $M > 0$ BE SUCH THAT $g, h \equiv 0$ OUTSIDE $[-M, M]$
(M EXISTS B/C g & h HAVE COMPACT SUPPORT)

<u>CASE 1</u> $\quad \underline{x \leq t - M}$

THEN $\quad x - t \leq -M$, SO $\quad g'(x-t) - h(x-t) = 0$

<u>CASE 2</u> $\quad \underline{x \geq M - t}$

THEN $\quad x + t \geq M$, SO $\quad g'(x+t) + h(x+t) = 0$

FINALLY, IF $\quad M - t \leq t - M \iff t \geq 2M$, SO IF $t \geq 2M$, THEN $(-\infty, t-M) \cup (M-t, \infty) = \mathbb{R}$, SO IF t IS LARGE ENOUGH, THIS INCLUDES ALL POSSIBLE CASES OF x.

IN ANY CASE, THIS IMPLIES THAT $\forall x$, THE FUNCTION INSIDE THE INTEGRAL IS 0, AND HENCE $k(t) - p(t) = 0$ FOR LARGE ENOUGH t, SO $\underline{k(t) = p(t)}$

ADDITIONAL PROBLEM 1

DEFINE $\tilde{g}(x) := \begin{cases} g(x) & x \geq 0 \\ g(-x) & x \leq 0 \end{cases}$ (THE EVENIFICATION OF g)

$\tilde{h}(x) := \begin{cases} h(x) & x \geq 0 \\ h(-x) & x \leq 0 \end{cases}$

AND SOLVE $\begin{cases} \tilde{U}_{tt} - \tilde{U}_{xx} = 0 & \text{IN } \mathbb{R} \times (0,\infty) \\ \tilde{U}(x,0) = \tilde{g}(x), \quad \tilde{U}_t(x,0) = \tilde{h}(x) & \text{ON } \mathbb{R} \times \{t=0\} \end{cases}$

BY D'ALEMBERT,

$$\boxed{\tilde{U}(x,t) = \frac{1}{2}\left(\tilde{g}(x-t) + \tilde{g}(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy}$$

AND DEFINE $U(x,t) = \tilde{U}(x,t)$ FOR $x > 0$

THEN $U(x,0) = \tilde{U}(x,0) = \tilde{g}(x) = g(x)$ IF $x > 0$ ✓

$U_t(x,0) = \tilde{U}_t(x,0) = \tilde{h}(x) = h(x)$ IF $x > 0$ ✓

$U_x(0,t) = \tilde{U}_x(0,t)$

BUT $\tilde{U}_x(x,t) = \frac{1}{2}\left(\tilde{g}'(x-t) + \tilde{g}'(x+t)\right) + \frac{1}{2}\left(\tilde{h}(x+t) - \tilde{h}(x-t)\right)$

SO $\tilde{U}_x(0,t) = \frac{1}{2}\left(\tilde{g}'(-t) + \tilde{g}'(t)\right) + \frac{1}{2}\left(\underbrace{\tilde{h}(t) - \tilde{h}(-t)}_{0 \text{ SINCE } \tilde{h} \text{ IS EVEN}}\right) = 0$

$\underbrace{\tilde{g}'(-t) + \tilde{g}'(t)}$: 0 SINCE $\tilde{g}$ IS EVEN AND HENCE $\tilde{g}'$ IS ODD

AND THEREFORE $U_x(0,t) = 0$

FINALLY, LET'S CLEAN UP THIS FORMULA A BIT:

CASE 1 $\quad x \geq t$

THEN $$\boxed{u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy}$$

CASE 2 $\quad x \leq t$

THEN $\quad \tilde{g}(x-t) = g(-(x-t)) = g(t-x)$

AND $$\int_{x-t}^{x+t} \tilde{h}(y)\,dy = \int_{x-t}^{0} \tilde{h}(y)\,dy + \int_{0}^{x+t} \tilde{h}(y)\,dy$$

$$= \int_{x-t}^{0} h(-y)\,dy + \int_{0}^{x+t} \tilde{h}(y)\,dy$$

$$= -\int_{t-x}^{0} h(y)\,dy + \int_{0}^{x+t} \tilde{h}(y)\,dy$$

$$= \int_{0}^{t-x} h(y)\,dy + \int_{0}^{x+t} h(y)\,dy$$

SO $$\boxed{u(x,t) = \frac{1}{2}\left(g(x+t) + g(t-x)\right) + \frac{1}{2}\left(\int_{0}^{x+t} h(y)\,dy + \int_{0}^{t-x} h(y)\,dy\right)}$$

$\boxed{\text{ADDITIONAL PROBLEM 2}}$

(a) $$V_{xx} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} U_{xx}\, ds$$

$U_{tt} = U_{xx}$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} U_{ss}\, ds$$

(IBP)

$$= \frac{-1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \left(e^{-\frac{s^2}{4t}}\right)_s U_s\, ds$$

$$= -\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \left(\frac{-s}{2t}\right) e^{-\frac{s^2}{4t}} U_s\, ds$$

$$= -\frac{1}{2t\sqrt{4\pi t}} \int_{-\infty}^{\infty} \left(s e^{-\frac{s^2}{4t}}\right)_s U\, ds$$

$$= -\frac{1}{2t\sqrt{4\pi t}} \int_{-\infty}^{\infty} \left(1 - \frac{s^2}{2t}\right) e^{-\frac{s^2}{4t}} U\, ds$$

ON THE OTHER HAND

$$V_t = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \left[\left(\frac{-1}{2t\sqrt{t}}\right) e^{-\frac{s^2}{4t}} + \left(\frac{1}{\sqrt{t}} \frac{s^2}{4t^2}\right) e^{-\frac{s^2}{4t}}\right] U\, ds$$

$$= -\frac{1}{2t\sqrt{4\pi t}} \int_{-\infty}^{\infty} \left(1 - \frac{s^2}{2t}\right) e^{-\frac{s^2}{4t}} U\, ds$$

$$= V_{xx}$$

(b) ON THE ONE HAND, WE KNOW BY PAGE 47 THAT

$$\tilde{V}(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-|x-y|^2}{4t}} g(y)\, dy$$

IS A BOUNDED SOLUTION OF $\begin{cases} \tilde{V}_t - \tilde{V}_{xx} = 0 \\ \tilde{V}(x,0) = g(x) \end{cases}$

ON THE OTHER HAND, WE ALSO KNOW V IS A SOLUTION

$$\begin{cases} V_t - V_{xx} = 0 \\ V(x,0) = g(x) \end{cases}$$

MOREOVER, V IS BOUNDED BECAUSE

$$|V(x,t)| \leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{4t}} \underbrace{|U(x,s)|}_{\leq C} \, ds \quad (\text{SINCE } U \text{ IS BDD})$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}}$$

$$= 1 \quad (\text{SEE LEMMA ON PAGE 46})$$

THEREFORE, BY UNIQUENESS OF THE CAUCHY PROBLEM (THEOREM 7 ON PAGE 59), WE GET

$V(x,t) = \tilde{V}(x,t)$, so

$$V(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-|x-y|^2}{4t}} g(y)\, dy \quad (*)$$

(c) ON THE ONE HAND

$$V(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} U(x,s)\, ds$$

B/C U IS EVEN IN t

$$= \frac{2}{\sqrt{4\pi t}} \int_{0}^{\infty} e^{-\frac{s^2}{4t}} U(x,s)\, ds$$

$\lambda := \frac{1}{4t}$

$$= \frac{1}{\sqrt{\pi t}} \int_{0}^{\infty} e^{-\lambda s^2} U(x,s)\, ds$$

ON THE OTHER HAND

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \overset{\text{POLAR COORDS}}{=} \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} \int_{\partial B(x,r)} e^{-\frac{r^2}{4t}} g(y)\, dS(y)\, dr$$

$$= \frac{1}{\cancel{\sqrt{4\pi t}}} \int_{0}^{\infty} e^{-\frac{r^2}{4t}} \underbrace{\left(\int_{\partial B(x,r)} g(y)\, dS(y) \right)}_{:= \cancel{2}\, G(r)} dr$$

$\lambda := \frac{1}{4t}$

$$= \frac{1}{\sqrt{\pi t}} \int_{0}^{\infty} e^{-\lambda r^2} G(r)\, dr$$

THEREFORE BY (*), WE GET

$$\cancel{\frac{1}{\sqrt{\pi t}}} \int_{0}^{\infty} e^{-\lambda s^2} U(x,s)\, ds = \cancel{\frac{1}{\sqrt{\pi t}}} \int_{0}^{\infty} e^{-\lambda r^2} G(r)\, dr$$

$$\int_{0}^{\infty} e^{-\lambda s^2} U(x,s)\, ds = \int_{0}^{\infty} e^{-\lambda r^2} G(r)\, dr \qquad (**)$$

(d) (**) EFFECTIVELY SAYS THAT

$$\mathcal{L}\, U(x) = \mathcal{L}\, G(x)$$

AND THEREFORE, BY UNIQUENESS OF THE LAPLACE TRANSFORM, WE GET

$$U(x,t) = G(x,t)$$

THAT IS, $\boxed{U(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right)}$

<u>NOTICE</u> THIS AGREES WITH D'ALEMBERT'S FORMULA WITH $h \equiv 0$.