Math 453 – Homework 9

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Friday, April 28, 2017

This assignment is due on **Friday, April 28, at 11:50 AM**

Reading: Section 2.4 (and Section 3.2)

Note: You may have noticed that we're into Chapter 3 by now, but I want to give you a little bit more time to digest the Chapter 3 material, since it's a bit tricky. Also, in this way you'll get more practice with the wave equation.

Chapter 2: 24, and the **two** additional problems below.

Hint for 24: For 24(a), you may assume there are no boundary terms (this follows from D'Alembert and because g and h have compact support). For 24(b), use D'Alembert's formula and that $(u_x)^2 - (u_t)^2 = (u_x - u_t)(u_x + u_t)$. Also, because g and h have compact support, there is some $M > 0$ where g and h are 0 outside of $[-M, M]$. You have to consider 3 cases: $x \leq t - M$, $x \geq M - t$ and $t - M \leq x \leq M - t$. That said, argue that if t is large enough, we actually have $M - t \leq t - M$, so we don't even have to consider the third case!

Additional Problem 1: Solve the following wave equation on the half-line:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (0, \infty) \times (0, \infty) \\ u(x, 0) = g(x), u_t(x, 0) = h(x) & \text{on } (0, \infty) \times \{t = 0\} \\ u_x(0, t) = 0 & \text{on } \{x = 0\} \times (0, \infty) \end{cases}$$

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Additional Problem 2: In this problem, we are going to solve the following wave equation (in one dimension) by turning it into... the heat equation!!! This proof can be generalized to odd dimensions.

Definition: If $\lambda > 0$ is fixed, the Laplace transform of $u = u(x,t)$ is:

$$\mathcal{L}u(x) = \int_0^\infty u(x,s)e^{-\lambda s^2}ds$$

Suppose that u is a bounded solution of

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x), u_t(x,0) = 0 & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

(here g is bounded). Extend u to negative times by writing, for $t < 0$,

$$u(x,t) = u(x,-t)$$

(a) Define

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^\infty e^{\frac{-s^2}{4t}} u(x,s)ds$$

Show directly that v must satisfy $v_t - v_{xx} = 0$ (You may assume there are no boundary terms).

It's also true (but you do not need to show this) that $v(x,0) = g(x)$ (at least in the limit as $t \to 0$), and therefore v satisfies

$$\begin{cases} v_t - v_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ v(x,0) = g(x) & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

(b) Solve for v. Make sure to verify that you can use uniqueness of solutions on all of $\mathbb{R}$!

(c) Using (b) the polar coordinates formula, deduce that, for $\lambda = \frac{1}{4t}$, that

$$\int_0^\infty u(x,s)e^{-\lambda s^2}ds = \int_0^\infty G(x,r)e^{-\lambda r^2}dr,$$

where $G(x,r) := \frac{1}{2}\int_{\partial B(x,r)} g(y)dS(y) = \frac{1}{2}(g(x+r)+g(x-r))$ (the latter because in one dimension, $B(x,r) = (x-r, x+r)$, whose boundary consists of the points $x \pm r$).

(d) Finally, use the following fact to solve for u:

Fact: The Laplace transform of a function is unique[1], that is:

$$\text{If } \mathcal{L}f(x) = \mathcal{L}g(x), \text{ then } f(x,t) \equiv g(x,t)$$

Compare with d'Alembert's formula for $h = 0$.

[1] If you're curious about the proof of this, see `http://www.ctr.maths.lu.se/media/MATC12/2013ht2013/uniqueness.pdf`