

Math 200 – Homework 9

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Monday, April 24, 2017

This assignment is due on **Monday, April 14, 2017** at 9:50 AM.

Reading: Section 7.4, 8.1, 8.2, 8.3. Ignore Example 7.4.1, the proof of Theorem 7.4.3, as well as Example 7.4.5 and the section “Application: Cardinality and Computability.” In lecture, we will just skim Chapter 8, but hopefully you’d be able to read a lot of it on your own. In section 8.1, you can ignore the section “N-ary relations and Relational Databases.” Example 8.1.6 is kinda cute, but it’ll make more sense once we cover Chapter 10. In section 8.2, ignore the section “The Transitive Closure of a Relation.” In Section 8.3, ignore examples 3, 4, 7, 9.

The second midterm is on **Friday, April 21, from 9 AM to 9:50 AM in class**. It will cover Sections 5.4 (Strong Induction) up to, and including Section 8.3 (Equivalence relations). I will send out an announcement with a study guide and a practice exam sometime before the week of the exam.

- **Section 7.4:** 21, 33, AP1, AP2, AP3
- **Section 8.1:** 11
- **Section 8.2:** 16, AP4
- **Section 8.3:** 21, 30, AP5

Note: For 7.4.21, just draw a picture of what the functions look like; think like the stock market!

(TURN PAGE for the Additional Problems)

Additional Problem 1:

- (a) Show that $(0, 1)$ and $(-\frac{\pi}{2}, \frac{\pi}{2})$ have the same cardinality (**Hint:** try a linear function from one to the other)
- (b) Show that $(-\frac{\pi}{2}, \frac{\pi}{2})$ and \mathbb{R} have the same cardinality.

Hint: Consider $f(x) = \tan(x)$. To show that f is one-to-one, use the fact that $f(x)$ is increasing (which follows from $f'(x) = \sec^2(x) > 0$), and the result from the additional problem from HW 8 (which says that if a function is increasing, then it is one-to-one). To show that f is onto, recall from the Intermediate Value Theorem from Calculus that if f is continuous on (a, b) , then f attains all the values between $f(a^+)$ and $f(b^-)$.

- (c) Use (a) and (b) and Theorem 7.4.1(c) to show that $(0, 1)$ and \mathbb{R} have the same cardinality. Surprising, isn't it??? At first sight, \mathbb{R} looks so much bigger than $(0, 1)$, but they actually have the same size!!!

Additional Problem 2: Show, using a picture (similar to what I did in lecture with \mathbb{Q} , or like the picture on page 433) that $\mathbb{Z} \times \mathbb{Z}$ is countable. (**Hint:** Start at $(0, 0)$, and think like a snail)

Additional Problem 3: Suppose the Grand Hotel Peyam is full with guests, and there are **two** busses with a (countably) infinite amount of people arriving. Show how to accomodate everyone while keeping the hotel full (this is just a spin-off on the example we did in class).

Additional Problem 4: Define the following relation on \mathbb{R} : $x \sim y \Leftrightarrow xy > 0$. Show that \sim is symmetric, transitive, but not reflexive (and therefore not an equivalence relation)

Hint: To show transitivity, argue in cases, depending on whether $x > 0$, $x = 0$ (can this happen?) or $x < 0$.

(TURN PAGE for Additional Problem 5)

Additional Problem 5: Define the following relation on \mathbb{R} : $x \sim y$ if and only if $x - y$ is rational (that is, $x - y = q$ for some $q \in \mathbb{Q}$).

- (a) Show that \sim is an equivalence relation.
- (b) Show that for every $a \in \mathbb{R}$, $[a]$ (the equivalence class of a) is countable.

Hint: Show that the function $f : \mathbb{Q} \rightarrow [a]$ defined by $f(q) = a + q$ is a bijection. (**Careful:** Here a is fixed, and we vary q . Also why is each $f(q)$ guaranteed to be in $[a]$? In other words, why is $f(q) \sim a$?)

- (c) Deduce using (b) that there are uncountably many (distinct) equivalence classes of the form $[a]$, where $a \in \mathbb{R}$.

Hint: Prove this by contradiction, assuming there are only countably many such classes, and use the fact that \mathbb{R} is a union (over all classes $[a]$) of all the elements in $[a]$.