HW 8 – SOLUTIONS

#19 (a) INTEGRATING $U_{xy} = 0$ WITH RESPECT TO x, WE GET

$$U_y = \tilde{G}(y) \quad \text{WHERE } \tilde{G} \text{ IS ARBITRARY}$$

AND INTEGRATING THIS WITH RESPECT TO y, WE GET

$$U(x,y) = F(x) + \int \tilde{G}(y)\,dy, \quad \text{WHERE } F \text{ IS ARBITRARY}$$

NOW IF G IS ARBITRARY, THEN WE CAN ALWAYS WRITE

G AS $\int \tilde{G}(y)\,dy$, NAMELY WITH $\tilde{G} = G_y$,

AND THEREFORE $U(x,y) = F(x) + G(y)$,

WHERE F AND G ARE ARBITRARY

(b)

$$U_x = U_\xi \frac{d\xi}{dx} + U_\eta \frac{d\eta}{dx} = U_\xi + U_\eta$$

$$U_t \quad U_\xi \frac{d\xi}{dt} + U_\eta \frac{d\eta}{dt} = U_\xi - U_\eta$$

HENCE

$$\begin{aligned} U_{xx} = (U_x)_\xi + (U_x)_\eta &= (U_\xi + U_\eta)_\xi + (U_\xi + U_\eta)_\eta \\ &= U_{\xi\xi} + U_{\eta\xi} + U_{\xi\eta} + U_{\eta\eta} \\ &= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta} \end{aligned}$$

AND

$$\begin{aligned} U_{tt} = (U_t)_\xi - (U_t)_\eta &= U_{\xi\xi} - U_{\eta\xi} - U_{\xi\eta} + U_{\eta\eta} \\ &= U_{\xi\xi} - 2U_{\eta\xi} + U_{\eta\eta} \end{aligned}$$

SO $\quad U_{tt} - U_{xx} = \cancel{U_{\xi\xi}} + 2U_{\xi\eta} + \cancel{U_{\eta\eta}}$

$$= \cancel{-U_{\xi\xi}} + 2U_{\xi\eta} - \cancel{U_{\eta\eta}}$$

$$= 4U_{\xi\eta}$$

AND THEREFORE $\quad U_{tt} - U_{xx} = 0 \iff 4U_{\xi\eta} = 0 \iff U_{\xi\eta} = 0$

(c) FROM (b), WE GET $\quad U_{tt} - U_{xx} = 0 \implies U_{\xi\eta} = 0$

AND FROM (a) WITH ξ REPLACING x AND η REPLACING y, WE GET

$$U(\xi,\eta) = F(\xi) + G(\eta)$$

THAT IS, $\quad U(x,t) = F(x+t) + G(x-t)$

NOW $\quad U(x,0) = F(x) + G(x) = g(x)$

$$U_t(x,t) = F'(x+t) - G'(x-t)$$

SO $\quad U_t(x,0) = F'(x) - G'(x) = h(x) \implies F(x) - G(x) = \int_0^x h(s)\,ds$

$$\implies F(x) - G(x) = \int_0^x h(s)\,ds + C$$

NOW SOLVING FOR F & G IN $\quad \begin{cases} F(x) + G(x) = g(x) \\ F(x) - G(x) = \int_0^x h(s)\,ds + C \end{cases}$

WE GET $\quad F(x) = \frac{1}{2}\left(g(x) + \int_0^x h(s)\,ds + C \right)$ (ADDING BOTH EQ)

$$G(x) = \frac{1}{2}\left(g(x) - \int_0^x h(s)\,ds - C \right)$$ (SUBTRACTING BOTH EQ)

AND SO: $$U(x,t) = F(x+t) + G(x-t)$$

$$= \frac{1}{2} g(x+t) + \frac{1}{2}\int_0^{x+t} h(s)\,ds + \cancel{\frac{c}{2}} + \frac{1}{2} g(x-t) - \frac{1}{2}\int_0^{x-t} h(s)\,ds - \cancel{\frac{c}{2}}$$

$$= \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\left(\int_0^{x+t} h(s)\,ds - \int_0^{x-t} h(s)\,ds\right)$$

$$U(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\left(\int_{x-t}^{x+t} h(s)\,ds\right)$$

WHICH IS D'ALEMBERT'S FORMULA

#21 (a) WE KNOW:

$$\begin{cases} E^1_t = B^3_{x_2} - B^2_{x_3} \\ E^2_t = B^1_{x_3} - B^3_{x_1} \\ E^3_t = B^2_{x_1} - B^1_{x_2} \end{cases} \qquad \begin{cases} B^1_t = E^2_{x_3} - E^3_{x_2} \\ B^2_t = E^3_{x_1} - E^1_{x_3} \\ B^3_t = E^1_{x_2} - E^2_{x_1} \end{cases}$$

1) THEN $$E^1_{tt} = (B^3_t)_{x_2} - (B^2_t)_{x_3} = E^1_{x_2x_2} \underbrace{- E^2_{x_1x_2} - E^3_{x_1x_3}} + E^1_{x_3x_3}$$

BUT $$\mathrm{DIV}(E)=0 \Rightarrow E^1_{x_1} + E^2_{x_2} + E^3_{x_3} = 0$$
$$\Rightarrow E^1_{x_1x_1} + E^2_{x_2x_1} + E^3_{x_3x_1} = 0$$
$$\Rightarrow E^2_{x_1x_2} + E^3_{x_1x_3} = -E^1_{x_1x_1}$$

HENCE $$E^1_{tt} = E^1_{x_1x_1} + E^1_{x_2x_2} + E^1_{x_3x_3} = \Delta E^1$$

SIMILARLY FOR E_2 AND E_3, HENCE $E_{tt} - \Delta E = 0$

2) AND $$B^1_{tt} = (E^2_t)_{x_3} - (E^3_t)_{x_2} = B^1_{x_3x_3} - B^3_{x_1x_3} - B^2_{x_1x_2} + B^1_{x_2x_2}$$

BUT $$\mathrm{DIV}(B) = 0 \Rightarrow B^1_{x_1} + B^2_{x_2} + B^3_{x_3} = 0$$
$$\Rightarrow B^1_{x_1x_1} + B^2_{x_1x_2} + B^3_{x_1x_3} = 0$$

$$\Rightarrow \quad B^2_{x_1x_2} + B^3_{x_1x_3} = -B^1_{x_1x_1}$$

HENCE $\quad B^1_{tt} = B^1_{x_1x_1} + B^1_{x_2x_2} + B^1_{x_3x_3} = \Delta B^1$

SIMILARLY FOR B^2 & B^3, HENCE $\quad B_{tt} - \Delta B = 0$

(b) NOTE SEE END OF THE SOLUTION FOR AN ALTERNATE DERIVATION.

WE KNOW:
$$\begin{cases} U^1_{tt} - \mu \Delta U^1 - (\lambda+\mu)\,(U^1_{x_1x_1} + U^2_{x_2x_1} + U^3_{x_3x_1}) = 0 \\ U^2_{tt} - \mu \Delta U^2 - (\lambda+\mu)\,(U^1_{x_1x_2} + U^2_{x_2x_2} + U^3_{x_3x_2}) = 0 \\ U^3_{tt} - \mu \Delta U^3 - (\lambda+\mu)\,(U^1_{x_1x_3} + U^2_{x_2x_3} + U^3_{x_3x_3}) = 0 \end{cases}$$

(1) THEN $\quad W_{tt} = (U^1_{x_1})_{tt} + (U^2_{x_2})_{tt} + (U^3_{x_3})_{tt}$

$$= \mu \Delta U^1_{x_1} + (\lambda+\mu)\,(U^1_{x_1x_1x_1} + U^2_{x_2x_1x_1} + U^3_{x_3x_1x_1})$$
$$+ \mu \Delta U^2_{x_2} + (\lambda+\mu)\,(U^1_{x_1x_2x_2} + U^2_{x_2x_2x_2} + U^3_{x_3x_2x_2})$$
$$+ \mu \Delta U^3_{x_3} + (\lambda+\mu)\,(U^1_{x_1x_3x_3} + U^2_{x_2x_3x_3} + U^3_{x_3x_3x_3})$$
$$= \mu \Delta U^1_{x_1} + (\lambda+\mu)\,((U^1_{x_1})_{x_1x_1} + (U^1_{x_1})_{x_2x_2} + (U^1_{x_1})_{x_3x_3})$$
$$+ \mu \Delta U^2_{x_2} + (\lambda+\mu)\,((U^2_{x_2})_{x_1x_1} + (U^2_{x_2})_{x_2x_2} + (U^2_{x_2})_{x_3x_3})$$
$$+ \mu \Delta U^3_{x_3} + (\lambda+\mu)\,((U^3_{x_3})_{x_1x_1} + (U^3_{x_3})_{x_2x_2} + (U^3_{x_3})_{x_3x_3})$$
$$= \mu \Delta U^1_{x_1} + (\lambda+\mu) \Delta U^1_{x_1} + \mu \Delta U^2_{x_2} + (\lambda+\mu)\Delta U^2_{x_2} + \mu \Delta U^3_{x_3}$$
$$-(\lambda+\mu)\Delta U^3_{x_3}$$
$$= (\lambda+2\mu)\,\Delta(U^1_{x_1} + U^2_{x_2} + U^3_{x_3}) = (\lambda+2\mu)\Delta W$$

THEREFORE $\quad W_{tt} - (\lambda + 2\mu)\Delta W = 0$

(2) $\quad W^1_{tt} = U^3_{x_2 tt} - U^2_{x_3 tt}$

$$= (U^3_{tt})_{x_2} - (U^2_{tt})_{x_3}$$

$$= (\mu \Delta U^3_{x_2} + (\lambda+\mu)(\cancel{U^1_{x_1 x_3 x_2}} + \cancel{U^2_{x_2 x_3 x_2}} + \cancel{U^3_{x_3 x_3 x_2}})$$

$$-(\mu \Delta U^2_{x_3} + (\lambda+\mu)(\cancel{U^1_{x_1 x_2 x_3}} + \cancel{U^2_{x_2 x_2 x_3}} + \cancel{U^3_{x_3 x_2 x_3}})$$

$$\Rightarrow \quad W^1_{tt} = \mu\, \Delta(U^3_{x_2} - U^2_{x_3}) = \mu \Delta W^1$$

HENCE $\quad W^1_{tt} - \mu \Delta W^1 = 0$, AND SIMILARLY FOR W^2, W^3

HENCE $\quad W_{tt} - \mu \Delta W = 0$

(#22) DIFFERENTIATING THE FIRST PDE WITH RESPECT TO t, WE GET

$$U_{tt} = -U_{xt} + d(V_t - U_t)$$

DIFFERENTIATING THE FIRST PDE WITH RESPECT TO x, WE GET

$$U_{xx} = -U_{xt} + d(V_x - U_x)$$

SUBTRACTING BOTH EQUATIONS, WE GET

$$U_{tt} - U_{xx} = \cancel{-U_{xt}} + d(V_t - U_t) + \cancel{U_{xt}} + d(U_x - V_x)$$
$$= d(V_t - U_t + U_x - V_x) \qquad (*)$$

BUT NOW ADDING BOTH PDE, WE GET

$$U_t + V_t + U_x - V_x = 0 \Rightarrow U_x - V_x = -U_t - V_t$$

HENCE (*) BECOMES:

$$U_{tt} - U_{xx} = d(\cancel{V_t} - U_t - U_t - \cancel{V_t}) = -2d\,U_t$$

$$\Rightarrow \quad U_{tt} + 2d\,U_t - U_{xx} = 0$$

IN A SYMMETRIC MANNER, WE OBTAIN $V_{tt} + 2d\,V_t - V_{xx} = 0$

NOTE

ALTERNATE SOLUTION TO #21(b) (THANKS, JANCE!)

(1) WE KNOW $\underline{U}_{tt} - \mu \Delta \underline{U} - (\lambda+\mu) D(\mathrm{DIV}(\underline{U})) = 0$

TAKING THE DIV $\mathrm{DIV}(\underline{U}_{tt}) - \mu\, \mathrm{DIV}(\Delta \underline{U}) - (\lambda+\mu)\, \mathrm{DIV}(D(\mathrm{DIV}(\underline{U})) = 0$

$$(\mathrm{DIV}(\underline{U}))_{tt} - \mu\, \Delta(\mathrm{DIV}(\underline{U})) - (\lambda+\mu)\, \Delta(\mathrm{DIV}(\underline{U})) = 0$$

NOTE $\mathrm{DIV}(\Delta \underline{U}) = (\Delta U^1)_{x_1} + (\Delta U^2)_{x_2} + (\Delta U^3)_{x_3}$
$= \Delta(U^1_{x_1} + U^2_{x_2} + U^3_{x_3}) = \Delta(\mathrm{DIV}(\underline{U}))$

$$W_{tt} - \mu \Delta W - (\lambda+\mu)\Delta W = 0$$

$$\boxed{W_{tt} - (\lambda + 2\mu)\Delta W = 0}$$

(2) THIS TIME TAKING CURL, WE GET

$$\mathrm{CURL}(\underline{U}_{tt}) - \mu\, \mathrm{CURL}(\Delta \underline{U}) - (\lambda+\mu)\, \underbrace{\mathrm{CURL}(D(\mathrm{DIV}(\underline{U}))}_{=0} = 0$$

$(\mathrm{CURL}(\underline{U}))_{tt} - \mu \Delta(\mathrm{CURL}(\underline{U})) = 0 \longrightarrow \mathrm{CURL}(\Delta \underline{U}) = (\Delta U^4_{x_2} - \Delta U^2_{x_1}, \dots)$

$$\boxed{W_{tt} - \mu \Delta W = 0}$$

$= \Delta(U^1_{x_2} - U^2_{x_1}, \dots)$
$= \Delta\, \mathrm{CURL}(\underline{U})$