HW 7 - SOLUTIONS

#1 LET $V(x,t) = U(x,t)\, e^{ct}$

THEN $V(x,0) = U(x,0)\, e^{c0} = g(x)$

AND $V_t = U_t\, e^{ct} + cU e^{ct}$

$= (-b \cdot DU)\, e^{ct} - \cancel{cUe^{ct}} \cancel{+cUe^{ct}}$

$= -b \cdot D(Ue^{ct})$

$= -b \cdot DV$

HENCE $V_t + b \cdot DV = 0$

SO V SOLVES $\begin{cases} V_t + b \cdot DV = 0 & \text{IN } \mathbb{R}^n \times (0,\infty) \\ V = g & \text{ON } \mathbb{R}^n \times \{t=0\} \end{cases}$

WHICH IS JUST THE TRANSPORT EQUATION, WHICH HAS SOLUTION

$V(x,t) = g(x - tb)$

HENCE $U(x,t)\, e^{ct} = g(x-tb)$

$$\boxed{U(x,t) = e^{-ct}\, g(x-tb)}$$

#18 LET $V = U_t$, THEN

$$V_{tt} = U_{ttt} = (U_{tt})_t = (\Delta U)_t = \Delta(U_t) = \Delta V$$

AND $V(x,0) = U_t(x,0) = h(x)$

AND $V_t(x,0) = U_{tt}(x,0) = (\Delta U)(x,0)$

BUT $U(x,0) = 0$, SO $\forall i$ $U_{x_i}(x,0) = 0$, SO $U_{x_i x_i}(x,0) = 0$
AND $\Delta U(x,0) = 0$,

THUS $V_t(x,0) = 0$, AND HENCE V SOLVES

$$\begin{cases} V_{tt} = \Delta V \\ V(x,0) = h(x), \quad V_t(x,0) = 0 \end{cases}$$

ADDITIONAL PROBLEM

LET $$E(t) = \int_0^{2\pi} (U(x,t))^2 \, dx$$

THEN $$E'(t) = \int_0^{2\pi} 2UU_t \, dx$$

$$= \int_0^{2\pi} -2UUU_x - 2UU_{xx} - 2UU_{xxxx}$$

$$= -2\int_0^{2\pi} U^2U_x + UU_{xx} + UU_{xxxx} \, dx$$

$$-2\left(\cancel{\left[\frac{U^3}{3}\right]_0^{2\pi}}^{\,0} + \cancel{\left[UU_x\right]_0^{2\pi}}^{\,0} - \int_0^{2\pi} (U_x)^2 + \cancel{\left[UU_{xxx}\right]_0^{2\pi}}^{\,0} - \int_0^{2\pi} U_x U_{xxx} \, dx \right)$$

BY 2π-PERIODICITY !

$$= 2\int_0^{2\pi} (U_x)^2 + 2\cancel{\left[U_x U_{xx}\right]_0^{2\pi}}^{\,0} - 2\int_0^{2\pi} (U_{xx})^2 dx$$

$$= 2\underbrace{\left(\int_0^{2\pi} (U_x)^2 - \int_0^{2\pi} (U_{xx})^2 \right)}_{\leq 0 \text{ (BY THE HINT)}}$$

HENCE $E'(t) \leq 0$ AND SO

$$E(t) \leq E(0) = \int_0^{2\pi} (\underbrace{U(x,0)}_{\equiv 0})^2 dx = 0$$

SO $0 \leq \int_0^{2\pi} (U(x,t))^2 dx \leq 0$, SO $\int_0^{2\pi} (U(x,t))^2 dx = 0$. HENCE $U \equiv 0$.