HW 6 - SOLUTIONS

#16 LET $u^\varepsilon(x,t) := u(x,t) + \varepsilon|x|^2$ AND SUPPOSE u^ε ATTAINS ITS MAXIMUM AT A POINT $(x,t) \in U_T$ (= INTERIOR $\cup$ TOP)

CASE 1 $\quad (x,t) \in U \times (0,T)$

THEN, BY THE TRUE SECOND DERIVATIVE TEST, $\Delta u^\varepsilon \leq 0$ AT (x,t) AND $u_t^\varepsilon = 0$ AT (x,t) (B/C (x,t) IS A CRITICAL POINT)

AND SO $\quad u_t^\varepsilon - \Delta u^\varepsilon \geq 0$ AT (x,t)

ON THE OTHER HAND,
$$
\begin{aligned}
u_t^\varepsilon - \Delta u^\varepsilon &= (u + \varepsilon|x|^2)_t - \Delta(u + \varepsilon|x|^2) \\
&= \underbrace{u_t - \Delta u} - 2\varepsilon N \\
&= 0 - 2\varepsilon N \\
&= -2\varepsilon N < 0
\end{aligned}
$$

SO IN PARTICULAR AT (x,t), $\quad u_t^\varepsilon - \Delta u^\varepsilon < 0 \quad \Rightarrow \Leftarrow$

CASE 2 $\quad (x,t) \in U \times \{t = T\}$

THEN BY THE TRUE SECOND DERIVATIVE TEST, WE STILL HAVE $\Delta u^\varepsilon \leq 0$ AT (x,t), BUT THIS TIME $u_t^\varepsilon \geq 0$ AND SO $u_t^\varepsilon - \Delta u^\varepsilon \geq 0$ AT (x,t) AND WE GET A SIMILAR CONTRADICTION.

SINCE u^ε CANNOT ATTAIN ITS MAXIMUM IN U_T, IT FOLLOWS THAT

$$\max_{\overline{U_T}} u^\varepsilon = \max_{\Gamma_T} u^\varepsilon$$

AND THEREFORE $\max_{\overline{U_T}} U \leq \max_{\overline{U_T}} U + \underbrace{\varepsilon |x|^2}_{\geq 0}$

$$= \max_{\overline{U_T}} U^\varepsilon$$

$$= \max_{\Gamma_T} U^\varepsilon = \max_{\Gamma_T} U + \varepsilon |x|^2$$

$$\leq \max_{\Gamma_T} U + \varepsilon \underbrace{\max_{\Gamma_T} |x|^2}_{:= C}$$

$$= \max_{\Gamma_T} U + \varepsilon C$$

AND LETTING $\varepsilon \to 0$, WE GET $\max_{\overline{U_T}} U \leq \max_{\Gamma_T} U$

BUT SINCE $\Gamma_T \subseteq \overline{U_T}$ WE ALSO HAVE $\max_{\Gamma_T} U \leq \max_{\overline{U_T}} U$

AND SO $\boxed{\max_{\overline{U_T}} U = \max_{\Gamma_T} U}$

#17) (a) THE ONLY DIFFERENCE WITH THE PROOF OF THE MEAN VALUE FORMULA IS THAT THIS TIME, IF WE LET

$$\varphi(r) = \frac{1}{r^N} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dy\, ds,$$

THEN WE HAVE

$$\varphi'(r) \geq \frac{1}{r^{N+1}} \iint_{E(r)} -4N \Delta u\, \psi - \frac{2N}{s} \sum_{i=1}^{N} u_{y_i} y_i \, dy\, ds$$

BECAUSE $u_s \leq \Delta u$, SO $-u_s \geq -\Delta u$

$$= 0$$

HENCE φ IS NONDECREASING, AND

$$\frac{1}{r^N} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dy\, ds = \varphi(r)$$

$$\geq \lim_{t \to 0} \varphi(t)$$

$$= \lim_{t \to 0} \frac{1}{t^N} \iint_{E(t)} \frac{|y|^2}{s^2}\, dy\, ds$$

$$= 4u(0,0)$$

$$\Rightarrow \quad u(0,0) \leq \frac{1}{4r^N} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dy\, ds$$

AND AFTER A TRANSLATION, WE HAVE

$$u(x,t) \leq \frac{1}{4r^N} \iint_{E(x,t;r)} u(y,s) \frac{|y-x|^2}{(s-t)^2}\, dy\, ds$$

(b) AGAIN, THE ONLY DIFFERENCE IS THAT INSTEAD OF HAVING EQUALITY THRU THE (BY (a)), WE HAVE

$$M = U(x_0,t_0) \leq \frac{1}{4T^N} \iint_{E(x_0,t_0,T)} U(y,s) \frac{|x_0-y|^2}{(t_0-s)^2} \leq M$$

AND EQUALITY <u>STILL</u> HOLDS ONLY IF $U \equiv M$ WITHIN $E(x_0,t_0;T)$

AND HENCE THE REST OF THE PROOF IS STILL THE SAME AND WE CAN STILL DEDUCE THAT $\displaystyle\max_{\overline{U_T}} U = \max_{\Gamma_T} U$

(c) LET $V = \varphi(U)$

THEN $V_t = \varphi'(U) U_t$

AND $\forall i$, $V_{x_i} = \varphi'(U) U_{x_i}$

$$V_{x_ix_i} = \underbrace{\varphi''(U)}_{\geq 0} \underbrace{(U_{x_i})^2}_{\geq 0} + \varphi'(U) U_{x_ix_i} \geq \varphi'(U) U_{x_ix_i}$$

WHENCE $\displaystyle \Delta V \geq \sum_{i=1}^{N} \varphi'(U) U_{x_ix_i} = \varphi'(U)\Delta U$

HENCE $V_t - \Delta V \leq \varphi'(U) U_t - \varphi'(U)\Delta U = \varphi'(U)\underbrace{(U_t - \Delta U)}_{=0} = 0$ ✓

(d) LET $\displaystyle V = |DU|^2 + (U_t)^2 = \sum_{j=1}^{N} (U_{x_j})^2 + (U_t)^2$

THEN $\displaystyle V_t = \sum_{j=1}^{N} 2(U_{x_j}) U_{tx_j} + 2 U_t U_{tt}$

AND $\forall i$ $\displaystyle V_{x_i} = \sum_{j=1}^{N} 2(U_{x_j})(U_{x_ix_j}) + 2(U_t)(U_{tx_i})$

$$V_{x_ix_i} = \sum_{j=1}^{N} 2\underbrace{(U_{x_ix_j})^2}_{\geq 0} + 2(U_{x_j})(U_{x_ix_ix_j}) + 2\underbrace{(U_{tx_i})^2}_{\geq 0} + 2(U_t)(U_{tx_ix_i})$$

$$V_{x_ix_i} \geq \sum_{j=1}^{N} 2(U_{x_j})(U_{x_ix_ix_j}) + 2(U_t)(U_{tx_ix_i})$$

WHENCE

$$\Delta V \geq \sum_{j=1}^{N} 2(U_{x_j})(\Delta U_{x_j}) + 2(U_t)(\Delta U_t)$$

HENCE $$V_t - \Delta V \leq \sum_{j=1}^{N} 2(U_{x_j})\underbrace{(U_t - \Delta U)}_{=0}{}_{x_j} + 2(U_t)\underbrace{(U_t - \Delta U)}_{=0}{}_t$$

$$V_t - \Delta V \leq 0 \quad \checkmark$$

$\boxed{\text{ADDITIONAL PROBLEM}}$

CONSIDER THE PDE
$$\begin{cases} U_t - U_{xx} = 0 & \text{IN } \mathbb{R} \times (0,\infty) \\ U(x,0) = \bar{f}(x) & \text{ON } \mathbb{R} \times \{t=0\} \end{cases}$$

BY OUR FORMULA FOR OUR INITIAL-VALUE PROBLEM (THEOREM 1 ON PAGE 47)

WE HAVE $$U(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\,\bar{f}(y)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \bar{f}(y)\,dy$$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} (-f(-y))\,dy + \int_{0}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy$$

CHANGE OF VARS $y' = -y$

$$= \int_{\infty}^{0} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x+y)^2}{4t}} f(y)\,dy + \int_{0}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy$$

$$U(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi t}} \left(e^{\frac{-(x-y)^2}{4t}} - e^{\frac{-(x+y)^2}{4t}} \right) f(y)\,dy \quad \left(\text{SINCE } \int_\infty^0 = -\int_0^\infty \right)$$

NOW U SOLVES $U_t - U_{xx} = 0$ IN $\mathbb{R} \times (0,\infty)$

SO IN PARTICULAR U SOLVES $U_t - U_{xx} = 0$ IN $(0,\infty) \times (0,\infty)$

ALSO ON $(0,\infty) \times \{t=0\}$, $U(x,0) = \bar{f}(x) = f(x)$ (SINCE $x>0$)

AND FINALLY, $U(0,t) = \int_0^\infty \frac{1}{\sqrt{4\pi t}} \left(e^{\frac{-(-y)^2}{4t}} - e^{\frac{-y^2}{4t}} \right) f(y)\,dy$

$$= \int_0^\infty \frac{1}{\sqrt{4\pi t}} \left(\cancel{e^{\frac{-y^2}{4t}}} - \cancel{e^{\frac{-y^2}{4t}}} \right) f(y)\,dy$$

$$= 0$$

SO U INDEED SOLVES
$$\begin{cases} U_t - U_{xx} = 0 & \text{IN } (0,\infty) \times (0,\infty) \\ U(x,0) = f(x) & \text{ON } (0,\infty) \times \{t=0\} \\ U(0,t) = 0 & \text{ON } \{x=0\} \times (0,\infty) \end{cases}$$