Math 453 – Homework 6

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Friday, April 7, 2017

This assignment is due on **Friday, April 7, at 11:50 AM** (after spring break)

Reading: Sections 2.3.2, 2.3.3 (skip (c)), 2.3.4.

Chapter 2: Additional Problem, 16, 17

Additional Problem: [Method of odd reflections] In the case $n = 1$, consider the following heat equation on the half-line

$$\begin{cases} u_t - u_{xx} = 0 \text{ in } (0,\infty) \times (0,\infty) \\ u(x,0) = f(x) \text{ on } (0,\infty) \times \{t = 0\} \\ u(0,t) = 0 \text{ on } \{x = 0\} \times (0,\infty) \end{cases}$$

where $f = f(x)$ is a given function. Show that

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) f(y) dy$$

Hint: Consider the odd extension $\overline{f}(x)$ of f defined by

$$\overline{f}(x) = \begin{cases} -f(-x) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ f(x) & \text{if } x > 0 \end{cases}$$

Then solve the PDE

$$\begin{cases} u_t - u_{xx} = 0 \text{ in } \mathbb{R} \times (0,\infty) \\ u(x,0) = \overline{f}(x) \text{ on } \mathbb{R} \times \{t = 0\} \end{cases}$$

Then write the solution you found in terms of f, and finally use a change of variables. Make sure to check that the solution you found satisfies the original PDE!

Hint for 16: I personally find it easier to define $u_\epsilon = u + \epsilon |x|^2$ and proceed like in Problem 4 in Chapter 2. Again, recall the true second derivative test: If a function $v(x,t)$ attains a maximum at an interior point (x,t), then $\Delta v \leq 0$ at (x,t) and $v_t = 0$ at (x,t) (because of critical point-ness). Beware that U_T also includes the top; in that case you still have $\Delta v \leq 0$, but this time $v_t \geq 0$.

Hint for 17: For parts (a) and (b), **Please don't rewrite the whole proof of the mean-value formula/maximum principle!** Just tell me what things you have to change in those proofs. For (c), recall that $\phi(s)$ is convex iff $\phi''(s) \geq 0$.