HW 5 - SOLUTIONS

#12 (a) LET $V(x,t) = U(\lambda x, \lambda^2 t)$

THEN $V_t = \lambda^2 U_t(\lambda x, \lambda^2 t)$

AND $V_{x_i} = \lambda U_{x_i}(\lambda x, \lambda^2 t)$,

SO $V_{x_i x_i} = \lambda^2 U_{x_i x_i}$

AND THEREFORE $\Delta V = \lambda^2 \Delta U$ (BY SUMMING OVER i)

AND HENCE $V_t = \lambda^2 U_t = \lambda^2 \Delta U = \Delta V$

SINCE U SOLVES THE HEAT EQUATION

(b) $V(x,t) = x \cdot DU + 2t\, U_t = \left(\sum_{j=1}^{N} x_j U_{x_j} \right) + 2t\, U_t$

$$V_t = \sum_{j=1}^{N} x_j (U_t)_{x_j} + 2U_t + 2t\, U_{tt} \qquad (*)$$

AND $$V_{x_i} = \left(\sum_{j=1}^{N} x_j U_{x_j} \right)_{x_i} + 2t (U_{x_i})_t$$

$$= U_{x_i} + \sum_{j=1}^{N} x_j U_{x_i x_j} + 2t (U_{x_i})_t$$

AND SIMILARLY

$$V_{x_i x_i} = U_{x_i x_i} + U_{x_i x_i} + \sum_{j=1}^{N} x_j U_{x_i x_i x_j} + 2t (U_{x_i x_i})_t$$

HENCE $$\Delta V = 2\Delta U + \sum_{j=1}^{N} x_j (\Delta U)_{x_j} + 2t (\Delta U)_t \qquad (**)$$

COMPARING $(*)$ AND $(**)$, WE NOTICE THAT THEY ARE EQUAL B/C $U_t = \Delta U$

#13 (a)

$$U_t = \left(V\left(\frac{x}{\sqrt{t}}\right)\right)_t = \left(-\frac{1}{2}\right)\frac{x}{t^{\frac{3}{2}}} \; V'\left(\frac{x}{\sqrt{t}}\right)$$

$$U_x = \frac{1}{\sqrt{t}} V'\left(\frac{x}{\sqrt{t}}\right), \quad U_{xx} = \left(\frac{1}{\sqrt{t}}\right)^2 V''\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{t} V''\left(\frac{x}{\sqrt{t}}\right)$$

HENCE $U_t = U_{xx} \Leftrightarrow -\frac{1}{2}\frac{x}{t^{\frac{3}{2}}} V' = \frac{1}{t} V''$

MULTIPLY BY t

$$\Leftrightarrow -\frac{x}{2\sqrt{t}} V' = V''$$

$$\Leftrightarrow V'' + \frac{x}{2\sqrt{t}} V' = 0$$

$$\Leftrightarrow V'' + \frac{z}{2} V' = 0 \qquad \left(z := \frac{x}{\sqrt{t}}\right) \qquad (*)$$

Now $V'' + \frac{z}{2} V' = 0$

$$\Leftrightarrow V'' = -\frac{z}{2} V'$$

$$\Leftrightarrow \frac{V''}{V'} = -\frac{z}{2}$$

$$\Leftrightarrow (\ln|V'|)' = -\frac{z}{2}$$

$$\Leftrightarrow \ln|V'| = -\frac{z^2}{4} + C$$

$$\Leftrightarrow |V'| = e^c e^{-\frac{z^2}{4}}$$

$$\Leftrightarrow V' = \underbrace{\pm e^c}_{C} e^{-\frac{z^2}{4}} \Leftrightarrow V' = Ce^{-\frac{z^2}{4}}$$

$$\Leftrightarrow V(z) = C\int_0^z e^{-\frac{s^2}{4}} ds + D$$

(b) NOTICE THAT IF U SOLVES $U_t = U_{xx}$, THEN SO DOES U_x, SO IN PARTICULAR,

$$U_x(x,t) = \left(V\left(\frac{x}{\sqrt{t}}\right)\right)_x = \frac{1}{\sqrt{t}} V'\left(\frac{x}{\sqrt{t}}\right) = \frac{C}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

SOLVES $U_t = U_{xx}$.

FINALLY, WE CHOOSE THE CONSTANT C SUCH THAT

$$\int_{-\infty}^{\infty} \frac{C}{\sqrt{t}} e^{-\frac{x^2}{4t}} \, dx = 1$$

$y = \frac{x}{2\sqrt{t}}$

$$\Downarrow$$

$$\int_{-\infty}^{\infty} \frac{C}{\cancel{\sqrt{t}}} e^{-y^2} \, 2\cancel{\sqrt{t}} \, dy = 1$$

$$2C \underbrace{\int_{-\infty}^{\infty} e^{-y^2} \, dy}_{\sqrt{\pi}} = 1$$

$$\Rightarrow \quad 2C\sqrt{\pi} = 1 \quad \Rightarrow \quad C = \frac{1}{2\sqrt{\pi}} = \frac{1}{\sqrt{4\pi}}$$

THEREFORE WE INDEED OBTAIN THE FUNDAMENTAL SOLUTION IN THE CASE $N=1$

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

#14 LET $V(x,t) = U(x,t)e^{ct}$ WHERE $U_t - \Delta U + cU = f$

THEN $V_t = (U_t)e^{ct} + cUe^{ct}$

$$= (\Delta U - \cancel{cU} + f)e^{ct} \cancel{+ cUe^{ct}}$$

$$= (\Delta U)e^{ct} + f(x,t)e^{ct}$$

$$= \Delta V + \underbrace{fe^{ct}}_{:= \tilde{f}}$$

HENCE V SOLVES $V_t - \Delta V = \tilde{f}$

MOREOVER, $V(x,0) = U(x,0)e^{0} = g(x)$, HENCE V SOLVES

$$\begin{cases} V_t - \Delta V = \tilde{f} & \text{IN } \mathbb{R}^N \times (0,t) \\ V(x,0) = g(x) & \text{ON } \mathbb{R}^N \times \{t=0\} \end{cases}$$

THEREFORE, BY FORMULA (17) ON PAGE 51, WE GET

$$V(x,t) = \int_{\mathbb{R}^N} \Phi(x-y,t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^N} \Phi(x-y,t-s)\, \tilde{f}(y,s)\, dy\, ds$$

$$\Rightarrow \boxed{U(x,t) = e^{-ct}\left(\int_{\mathbb{R}^N} \Phi(x-y)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^N} \Phi(x-y,t-s)\, f(y,s)\, e^{cs}\, dy\, ds \right)}$$

ADDITIONAL PROBLEM SUPPOSE $U \in \mathcal{A}$ MINIMIZES

$$I[U] = \int_W \frac{1}{2}|DU|^2 - F(U)$$

LET $V \in C_c^\infty(W)$ BE ARBITRARY AND CONSIDER:

$$i(\tau) = I[U+\tau V] = \int_W \frac{1}{2}|DU+\tau DV|^2 - F(U+\tau V)$$

THEN:

$$i'(\tau) = \int_W (DU+\tau DV)\cdot DV - F'(U+\tau V)\,V$$

$$= \int_W (DU + \tau DV)\cdot DV - f(U+\tau V)\,V$$

HENCE, SINCE V IS A MINIMIZER, WE GET:

$$0 = i'(0) = \int_W DU \cdot DV - f(U)\,V$$

IBP

$$= \int_{\partial W} \cancel{\left(\frac{\partial U}{\partial \nu}\right)\underbrace{V}_{0 \text{ ON } \partial W}}^{0} - \int_W (\Delta U)\,V - \int_W f(U)\,V$$

$$= \int_W (-\Delta U - f(U))\,V = 0$$

BUT SINCE V WAS ARBITRARY, WE GET $-\Delta U - f(U) = 0$

SO $\boxed{-\Delta U = f(U)}$