HW 4 - SOLUTION

#4 LET $U_\varepsilon(x) = U(x) + \varepsilon |x|^2$

STEP 1

SUPPOSE BY CONTRADICTION THAT U_ε ATTAINS ITS MAXIMUM OVER $\overline{W}$ AT AN INTERIOR POINT $x_0 \in W$.

THEN, BY THE FACT DISCUSSED ON THE HANDOUT ON THE TRUE SECOND-DERIVATIVE TEST, WE HAVE

$$\Delta U^\varepsilon(x_0) \leq 0$$

BUT $\Delta U^\varepsilon = \Delta U + \Delta \varepsilon |x|^2 = \varepsilon \Delta |x|^2$ (SINCE $\Delta U = 0$)

NOW $\Delta |x|^2 = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} (x_j)^2 \right)_{x_i x_i} = \sum_{i=1}^{N} 2 = 2N$

HENCE $\Delta U^\varepsilon = 2N\varepsilon$

NOW AT $x = x_0$, WE HAVE $\underbrace{\Delta U^\varepsilon(x_0)}_{\leq 0} = 2N\varepsilon > 0$

WHICH IS A CONTRADICTION.

HENCE U^ε CANNOT ATTAIN ITS MAXIMUM AT AN INTERIOR POINT.

AND HENCE $\max_{\overline{W}} U^\varepsilon(x) = \max_{\partial W} U^\varepsilon(x)$

STEP 2

NOW $\max_{\overline{W}} U(x) \leq \max_{\overline{W}} U(x) + \underbrace{\varepsilon |x|^2}_{\geq 0}$

$$= \max_{\overline{W}} U^\varepsilon(x)$$

$$= \max_{\partial W} U^{\varepsilon}(x) \quad (\text{BY STEP 1})$$

$$= \max_{\partial W} U(x) + \varepsilon |x|^2$$

$\max(f+g) \leq \max(f) + \max(g)$

$$\leq \max_{\partial W} U(x) + \max_{\partial W} \varepsilon |x|^2$$

$$= \left(\max_{\partial W} U(x) \right) + \varepsilon \underbrace{\max_{\partial W} |x|^2}_{\leq C}$$

$$\leq \left(\max_{\partial W} U(x) \right) + \varepsilon C$$

HENCE $\max_{\overline{W}} U(x) \leq \max_{\partial W} U(x) + \varepsilon C$

AND LETTING $\varepsilon \to 0$ WE GET $\max_{\overline{W}} U(x) \leq \max_{\partial W} U(x)$

AND WE ALSO HAVE $\max_{\partial W} U(x) \leq \max_{\overline{W}} U(x)$ BECAUSE $\partial W \subseteq \overline{W}$

AND THEREFORE $\max_{\overline{W}} U(x) = \max_{\partial W} U(x)$

#5 (a) LET $\varphi(r) = ⨍_{\partial B(x,r)} V(y)\, dS(y)$

THEN, AS IN THE PROOF OF THE MVF, WE HAVE

$$\varphi'(r) = \frac{r}{N} ⨍_{B(x,r)} \underbrace{\Delta V(y)}_{\geq 0}\, dy \geq 0$$

HENCE φ IS NONDECREASING AND

$$\varphi(r) \geq \lim_{t \to 0} \varphi(t) = \lim_{t\to 0} ⨍_{\partial B(x,t)} V(y)\, dS(y) = V(x)$$

SO $$⨍_{\partial B(x,r)} V(y)\, dS(y) \geq V(x) \qquad (*)$$

FINALLY, EMPLOYING $(*)$ AND THE POLAR COORDINATE FORMULA, WE GET:

$$⨍_{B(x,r)} V(y)\, dy = \frac{1}{\alpha(N) r^N} \int_{B(x,r)} V(y)\, dy$$

$$= \frac{1}{\alpha(N) r^N} \int_0^r \left(\int_{\partial B(x,t)} V(y)\, dS(y) \right) dt$$

$$\geq \frac{1}{\cancel{\alpha(N)} r^N} \int_0^r V(x)\, N \cancel{\alpha(N)}\, t^{N-1}\, dt$$

$$= \frac{N V(x)}{r^N} \left[\frac{t^N}{N} \right]_0^r = \frac{\cancel{N} V(x)}{\cancel{r^N}} \frac{\cancel{r^N}}{\cancel{N}}$$

$$= V(x)$$

SO $$⨍_{B(x,r)} V(y)\, dy \geq V(x)$$

(b) THE PROOF FOLLOWS EXACTLY LIKE THE ONE OF THE STRONG MAXIMUM PRINCIPLE, EXCEPT THAT INSTEAD OF HAVING

$$M = U(x_0) = \fint_{B(x,r)} U(y)\,dy \le M, \quad \text{WE HAVE}$$

$$M = U(x_0) \le \fint_{B(x,r)} U(y)\,dy \le M$$

FROM WHICH IT STILL FOLLOWS THAT $\fint_{B(x,r)} U(y)\,dy = M$

(c) $V(x) = \varphi(U(x))$

$$V_{x_i} = \varphi'(U)\, U_{x_i}$$

$$V_{x_i x_i} = \underbrace{\varphi''(U)}_{\geq 0}\underbrace{(U_{x_i})^2}_{\geq 0} + \varphi'(U)\, U_{x_i x_i} \geq \varphi'(U)\, U_{x_i x_i}$$

HENCE $$\Delta V = \sum_{i=1}^{N} V_{x_i x_i} \geq \sum_{i=1}^{N} \varphi'(U)\, U_{x_i x_i} = \varphi'(U) \sum_{i=1}^{N} U_{x_i x_i}$$

$$= \varphi'(U)\, \Delta U = 0$$

HENCE $\Delta V \geq 0$ SO $-\Delta V \leq 0$

(d) $$V(x) = |DU|^2 = \sum_{j=1}^{N} (U_{x_j})^2$$

THEN $$V_{x_i} = \sum_{j=1}^{N} 2\, U_{x_j} U_{x_j x_i}$$

$$V_{x_i x_i} = \sum_{j=1}^{N} 2\, U_{x_j x_i} U_{x_j x_i} + 2\, U_{x_j} U_{x_j x_i x_i}$$

$$= \sum_{j=1}^{N} 2\,(U_{x_i x_j})^2 + 2\, U_{x_j} U_{x_j x_i x_i} \geq 2 \sum_{j=1}^{N} U_{x_j} U_{x_j x_i x_i}$$

so $$\Delta V = \sum_{i=1}^{N} V_{x_i x_i}$$

$$\geq 2 \sum_{i=1}^{N} \sum_{j=1}^{N} U_{x_j} U_{x_j x_i x_i}$$

$$= 2 \sum_{j=1}^{N} \sum_{i=1}^{N} U_{x_j} U_{x_j x_i x_i}$$

$$= 2 \sum_{j=1}^{N} U_{x_j} \sum_{i=1}^{N} U_{x_j x_i x_i}$$

$$= 2 \sum_{j=1}^{N} U_{x_j} \left(\sum_{i=1}^{N} U_{x_i x_i} \right)_{x_j}$$

$$= 2 \sum_{j=1}^{N} U_{x_j} \Big(\underbrace{\Delta U}_{=0} \Big)_{x_j}$$

$$= 0$$

HENCE $\Delta V \geq 0$, so $-\Delta V \leq 0$

#6

STEP 1 FIRST OF ALL, LET'S VERIFY THAT

$$-\Delta\left(u + \frac{|x|^2}{2N}\lambda\right) \leq 0 \quad \text{FOR } \lambda := \max_{\overline{W}} |f|$$

$$-\Delta\left(u + \frac{|x|^2}{2N}\lambda\right) = -\Delta u - \frac{\lambda}{2N}\Delta(|x|^2)$$

BY #4

$$= f - \frac{\lambda}{2N}(2N)$$

$$= f - \lambda$$

$$= f - \max_{\overline{W}} |f|$$

$$\leq 0 \quad (\text{BY DEF OF } \max)$$

STEP 2 THEREFORE, IT FOLLOWS THAT u IS SUBHARMONIC, AND HENCE BY #5 (b), WE HAVE

$$\max_{\overline{W}} u + \frac{|x|^2}{2N}\lambda = \max_{\partial W} \underbrace{u}_{g} + \frac{|x|^2}{2N}\lambda$$

HENCE WE GET:

$$\max_{\overline{W}} u \leq \max_{\overline{W}}\left(u + \underbrace{\frac{|x|^2}{2N}\lambda}_{\geq 0}\right)$$

$$= \max_{\partial W} g + \frac{|x|^2}{2N}\lambda$$

$$\leq \max_{\partial W} g + \max_{\partial W} \frac{|x|^2}{2N}\lambda$$

$$\leq \max_{\partial W} |g| + \underbrace{\max_{\partial W} \frac{|x|^2}{2N}}_{\leq C' \text{ (SINCE } \partial W \text{ IS BOUNDED)}} \cdot$$

$$\leq \max_{\partial W} |g| + C' \max_{\overline{W}} |f|$$

$$\leq C\left(\max_{\partial W} |g| + \max_{\overline{W}} |f| \right) \qquad (\text{FOR } C = \max(1, C'))$$

HENCE $\displaystyle \max_{\overline{W}} U \leq C\left(\max_{\partial W} |g| + \max_{\overline{W}} |f| \right) \quad (*)$

STEP 3 ON THE OTHER HAND, NOTICE THAT $V := -U$ SATISFIES

$$\begin{cases} -\Delta V = -f & \text{IN } W \\ V = -g & \text{ON } \partial W \end{cases}$$

SO BY $(*)$ APPLIED TO V, WE GET

$$\max_{\overline{W}} V \leq C\left(\max_{\partial W} |-g| + \max_{W} |-f| \right)$$

$$\max_{\overline{W}} -U \leq C\left(\max_{\partial W} |g| + \max_{W} |f| \right) \quad (**)$$

STEP 4 COMBINING $(*)$ AND $(**)$ AND USING $|U| = \pm U$, WE GET

$$\max_{\overline{W}} |U| \leq C\left(\max_{\partial W} |g| + \max_{\overline{W}} |f| \right)$$