# HW 3 - SOLUTIONS

#3) 1) LET $\varphi(r) = \fint_{\partial B(0,r)} u(y)\, ds(y)$

THEN, EXACTLY LIKE IN THE PROOF OF THE MEAN-VALUE FORMULAS, WE ARRIVE AT THE CONCLUSION:

$$\varphi'(r) = \frac{r}{N} \fint_{B(0,r)} \Delta u(y)\, dy$$

$$= \frac{r}{N} \fint_{B(0,r)} (-f(y))\, dy$$

$$= -\frac{r}{N} \frac{1}{|B(0,r)|} \int_{B(0,r)} f(y)\, dy$$

$$= -\frac{r}{N} \frac{1}{\alpha(N) r^N} \int_{B(0,r)} f(y)\, dy$$

$$\varphi'(r) = \frac{-1}{N\alpha(N) r^{N-1}} \int_{B(0,r)} f(y)\, dy$$

2) NOW BY THE FUNDAMENTAL THEOREM OF CALCULUS, WE HAVE:

$$\varphi(r) - \varphi(0) = \int_0^r \varphi'(s)\, ds$$

$$\fint_{\partial B(0,r)} \underbrace{u(y)}_{g(y)}\, ds(y) - \underbrace{\fint_{\partial B(0,0)} u(y)\, ds(y)}_{u(0)} = -\frac{1}{N\alpha(N)} \int_0^r \frac{1}{s^{N-1}} \int_{B(0,s)} f(y)\, dy\, ds$$

$$= -\frac{1}{N\alpha(N)}\left(\left[\frac{-1}{(N-2)s^{N-2}}\int_{B(0,s)} f(y)\,dy\right]_{s=0}^{s=r} + \int_0^r \frac{1}{(N-2)s^{N-2}}\int_{\partial B(0,s)} f(y)\,dy\,ds\right)$$

(REGULAR IBP AND THEOREM 4 (ii) ON PAGE 7/2)

$$= \left(\frac{1}{N\alpha(N)}\right)\left(\frac{1}{(N-2)r^{N-2}}\right)\int_{B(0,r)} f(y)\,dy$$

$$+ \frac{1}{N\alpha(N)}\int_0^r \frac{1}{N-2}\left(\int_{\partial B(0,s)} \frac{f(y)}{s^{N-2}}\,dy\right)ds$$

$$= \frac{1}{N(N-2)\alpha(N)r^{N-2}}\int_{B(0,r)} f(y)\,dy - \frac{1}{N(N-2)\alpha(N)}\underbrace{\int_0^r\int_{\partial B(0,s)} \frac{f(y)}{|y|^{N-2}}}_{\int_{B(0,r)} \frac{f(y)}{|y|^{N-2}}}$$

(BY POLAR COORDINATES)

$$= \frac{1}{N(N-2)\alpha(N)}\int_{B(0,r)}\left(\frac{1}{r^{N-2}} - \frac{1}{|y|^{N-2}}\right) f(y)\,dy$$

3) HENCE $\displaystyle ⨍_{\partial B(0,r)} g(y)\,dS(y) - U(0) = \frac{1}{N(N-2)\alpha(N)}\int_{B(0,r)}\left(\frac{1}{r^{N-2}} - \frac{1}{|y|^{N-2}}\right) f(y)\,dy$

$$\Rightarrow \quad U(0) = ⨍_{\partial B(0,r)} g(y)\,dS(y) + \frac{1}{N(N-2)\alpha(N)}\int_{B(0,r)}\left(\frac{1}{|y|^{N-2}} - \frac{1}{r^{N-2}}\right) f(y)\,dy$$

PROBLEM 1

$$\left| \frac{1}{|\partial B(x,\varepsilon)|} \int_{\partial B(x,\varepsilon)} f(y)\, ds(y) - f(x) \right|$$

$$= \left| \frac{1}{|\partial B(x,\varepsilon)|} \int_{\partial B(x,\varepsilon)} f(y)\, ds(y) - \frac{1}{|\partial B(x,\varepsilon)|} \int_{\partial B(x,\varepsilon)} f(x)\, ds(y) \right|$$

$$= \frac{1}{|\partial B(x,\varepsilon)|} \left| \int_{\partial B(x,\varepsilon)} f(y) - f(x) \, ds(y) \right|$$

$$\leq \frac{1}{|\partial B(x,\varepsilon)|} \int_{\partial B(x,\varepsilon)} |f(y) - f(x)| \, ds(y)$$

$$\leq \frac{1}{|\partial B(x,\varepsilon)|} \sup_{y \in \partial B(x,\varepsilon)} |f(y) - f(x)| \int_{\partial B(x,\varepsilon)} ds(y)$$

$$= \frac{1}{\cancel{|\partial B(x,\varepsilon)|}} \sup_{y \in \partial B(x,\varepsilon)} |f(y) - f(x)| \cancel{|\partial B(x,\varepsilon)|}$$

$$\leq \sup_{y \in \partial B(x,\varepsilon)} |f(y) - f(x)| \longrightarrow 0 \text{ AS } \varepsilon \to 0$$

THE LAST STEP FOLLOWS SINCE $f$ IS CONTINUOUS AT $x$

PROBLEM 2

(a) $$V(r) = |B(x,r)|$$

$$= \int_{B(x,r)} 1 \, dy$$

$$z = \frac{y-x}{r}$$

$$dz = \frac{1}{r^N} dy \Rightarrow dy = r^N dz$$

$$= \int_{B(0,1)} 1 (r^N dz)$$

$$= r^N \int_{B(0,1)} 1 \, dz$$

$$= r^N |B(0,1)|$$

$$V(r) = \alpha(N) r^N$$

(b) $$V(r) = \int_{B(x,r)} 1 \, dy$$

$$= \int_0^r \left( \int_{\partial B(x,t)} 1 \, ds(y) \right) dt \qquad = r^{N-1} dz$$

$$V(r) = \int_0^r A(t) \, dt$$

(c) THEREFORE $V'(r) = A(r)$, so

$$A(r) = (\alpha(N) r^N)' = N \alpha(N) r^{N-1}$$