Math 453 – Homework 3

Peyam Tabrizian

Friday, February 24, 2017

This assignment is due on **Friday, February 24, at 10:50 AM**.

Reading: Sections 2.2.1, 2.2.2, and 2.2.3. Although we'll cover the maximum principle in lecture, I won't assign problems on it until Homework 4 (otherwise this problem set will be too heavy).

Chapter 2: 3, and the **two** problems below

Hint for 3: Mimic the proof of the mean-value formula, but with $x = 0$. Then, given r, integrate from 0 to r. At some point, you should be stuck with a term of the form:

$$\int_0^r \frac{1}{s^{n-1}} \int_{B(0,s)} f(y) dy ds$$

To handle this term, let $h(s) = \int_{B(0,s)} f(y)dy$ and then integrate by parts (just the regular 1D-Calculus version, not the one that we learned in class) with respect to s. You may assume, without proof, that the term for $s = 0$ is zero. You would also need to use the result of Theorem 4(ii) on page 712 to find $h'(s)$. Finally, at some point you may need to write s as $|y|$ and use the polar coordinate formula.

Problem 1: Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then, as $\epsilon \to 0$, we have

$$\frac{1}{|\partial B(x,\epsilon)|} \int_{\partial B(x,\epsilon)} f(y) dS(y) \to f(x)$$

Hint: Write the fixed number $f(x)$ as an integral over $\partial B(x,\epsilon)$. Also use the fact that, f is uniformly continuous on, say, $\overline{B(x,1)}$ (the closed ball of radius 1 centered at x, which is compact).

Problem 2: This problem is a fun application of the change-of-variables formula and the polar coordinates formula! For this problem, you are **not** allowed to use the volume/surface area formulas I talked about in lecture because the point of this problem is to derive them from scratch!

(a) Recall that $\alpha(n)$ is the valume of the unit ball $B(0, 1)$ in $\mathbb{R}^n$. Use this fact and a change-of-variables to find a formula for $V(r)$, the volume of $B(x, r)$, in terms of r and $\alpha(n)$.

(b) Let $A(r)$ be the surface area of $\partial B(x, r)$. Use the polar coordinates formula to find a formula involving integrals that relates $V(r)$ and $A(r)$.

(c) Differentiate the integral in (b) with respect to r to find a formula for $A(r)$ in terms of r and $\alpha(n)$. Neat, huh? This explains precisely why $A(r)$ is a derivative of $V(r)$!

Hint: How can you write the volume/surface area of a region in terms of integrals?