HW 2 – SOLUTIONS

#1 LET $\varepsilon > 0$ BE GIVEN, THEN SINCE f IS UNIFORMLY CONTINUOUS (SINCE f IS CONTINUOUS AND HAS COMPACT SUPPORT), WE KNOW THAT $\exists\, \delta > 0$ SUCH THAT FOR ALL x & $x' \in \mathbb{R}^N$, IF

$$|x - x'| < \delta \quad , \text{ THEN } \quad |f(x) - f(x')| < \varepsilon$$

NOW LET $\delta > 0$ BE AS ABOVE, AND LET x & $x' \in \mathbb{R}^N$ BE GIVEN

THEN $\left| (f * g)(x) - (f * g)(x') \right|$

$$= \left| \int_{\mathbb{R}^N} g(y)\, f(x-y) - \int_{\mathbb{R}^N} g(y)\, f(x'-y) \right|$$

$$= \left| \int_{\mathbb{R}^N} g(y) \left(f(x-y) - f(x'-y) \right) \right|$$

$$\leq \int_{\mathbb{R}^N} |g(y)|\, |f(x-y) - f(x'-y)|\, dy$$

BUT NOW, NOTICE THAT $|(x-y) - (x'-y)| = |x - x'| < \delta$, THEREFORE BY UNIFORM CONTINUITY OF f, WE HAVE

$$|f(x-y) - f(x'-y)| < \varepsilon$$

AND THEREFORE,

$$|(f*g)(x) - (f*g)(x')| \leq \int_{\mathbb{R}^N} |g(y)|\, \varepsilon\, dy = \varepsilon \underbrace{\int_{\mathbb{R}^N} |g(y)|\, dy}_{< \infty}$$

So $\quad |(f*g)(x) - (f*g)(x')| \leq C\varepsilon$

#2 (a) PROOF OF THEOREM 2 :

APPLYING THEOREM 1 TO UV GIVES :

$$\int_W (UV)_{x_i} = \int_{\partial W} UV \nu^i$$

$$\int_W U_{x_i} V + \int_W U V_{x_i} = \int_{\partial W} UV \nu^i$$

$$\int_W U_{x_i} V = -\int_W U V_{x_i} + \int_{\partial W} UV \nu^i$$

(b) PROOF OF THEOREM 3 (i)

REPLACING U WITH U_{x_i} IN THEOREM 2 AND $V \equiv 1$, WE GET

$$\int_W U_{x_i x_i} = -\int_W U_{x_i} \cancel{(1)_{x_i}}^{\,0} + \int_{\partial W} U_{x_i} \nu^i$$

SUMMING OVER i, WE HAVE

$$\int_W \sum_{i=1}^{N} U_{x_i x_i} = \int_{\partial W} \sum_{i=1}^{N} U_{x_i} \nu^i$$

$$\int_W \Delta U = \int_{\partial W} DU \cdot \nu = \int_{\partial W} \frac{\partial U}{\partial \nu} \quad \left(\text{BY DEFINITION OF } \frac{\partial U}{\partial \nu}\right)$$

(c) PROOF OF THEOREM 3(ii)

REPLACING V WITH V_{x_i} IN THEOREM 2, WE GET

$$\int_W U_{x_i} V_{x_i} = -\int_W U V_{x_i x_i} + \int_{\partial W} U V_{x_i} \nu^i$$

SUMMING OVER i, WE HAVE

$$\int_W \sum_{i=1}^{N} U_{x_i} V_{x_i} = -\int_W U \sum_{i=1}^{N} V_{x_i x_i} + \int_{\partial W} U \sum_{i=1}^{N} V_{x_i} \nu^i$$

$$\int_W DU \cdot DV = -\int_W U \Delta V + \int_{\partial W} U \, DV \cdot \nu$$

$$\int_W DV \cdot DU = -\int_W U \Delta V + \int_{\partial W} U \frac{\partial V}{\partial \nu} \quad \left(\text{BY DEF OF } \frac{\partial V}{\partial \nu}\right)$$

(d) PROOF OF THEOREM 3(iii)

WE HAVE

$$\int_W DV \cdot DU = -\int_W U \Delta V + \int_{\partial W} \frac{\partial V}{\partial \nu} U$$

$$\int_W DU \cdot DV = -\int_W V \Delta U + \int_{\partial W} \frac{\partial U}{\partial \nu} V$$

SUBTRACT THE FIRST EQUATION FROM THE SECOND, AND GET

$$\cancel{\int_W \nabla U \cdot \nabla V} - \cancel{\int_W \nabla V \cdot \nabla U} = -\int_W V \Delta U + \int_W U \Delta V$$

$$+ \int_{\partial W} \frac{\partial U}{\partial \nu} V - \int_{\partial W} \frac{\partial V}{\partial \nu} U$$

$$\Rightarrow \int_W U \Delta V - V \Delta U = \int_{\partial W} U \frac{\partial V}{\partial \nu} - V \frac{\partial U}{\partial \nu}$$ ✓

#3 MULTIPLYING THE FIRST EQUATION BY V AND THE SECOND EQUATION BY U, ADDING THEM UP, AND INTEGRATING ON $B(0,r)$, WE GET:

$$\int_{B(0,r)} (\Delta U) V - (\Delta V) U = \int_{B(0,r)} V^2 + U^2$$

NOW BY THEOREM 3 (iii) ON PAGE 712 (GREEN'S FORMULA 3), OR BY INTEGRATING BY PARTS TWICE, WE OBTAIN THAT:

$$\underbrace{\int_{\partial B(0,r)} \left(\frac{\partial U}{\partial \nu}\right) V - \left(\frac{\partial V}{\partial \nu}\right) U}_{A} = \underbrace{\int_{B(0,r)} V^2 + U^2}_{B} \quad (*)$$

NOW, AS $r \to \infty$, $B \to \int_{\mathbb{R}^N} U^2 + V^2$

AND NOW, $|A| \leq \int_{\partial B(0,r)} \left| \frac{\partial U}{\partial \nu} V - \frac{\partial V}{\partial \nu} U \right|$

$$\leq \int_{\partial B(0,r)} \left|\frac{\partial U}{\partial \nu}\right| |V| + \left|\frac{\partial V}{\partial \nu}\right| |U|$$

BUT $\left|\frac{\partial U}{\partial \nu}\right| = |DU \cdot \nu| \leq |DU| \underbrace{|\nu|}_{=1} = |DU| \leq C$ (BY ASSUMPTION)

AND $|V| \leq \frac{C}{|x|^N}$ (BY ASSUMPTION)

AND SIMILARLY FOR THE OTHER TERMS, SO

$$|A| \leq \int_{\partial B(0,r)} \frac{C}{|x|^N} + \frac{C}{|x|^N} = C \int_{\partial B(0,r)} \frac{1}{|x|^N}$$

NOW ON $\partial B(0,r)$, $|x| = r$, SO

$$C \int_{\partial B(0,r)} \frac{1}{|x|^N} = C \int_{\partial B(0,r)} \frac{1}{r^N} = \frac{C}{r^N} \int_{\partial B(0,r)} 1$$

$$= \frac{C}{r^N} |\partial B(0,r)| = \frac{C}{r^N} N\alpha(N) r^{N-1} = \frac{C N\alpha(N)}{r} \to 0 \text{ AS } r \to \infty$$

HENCE, AS $r \to \infty$, $A \to 0$, AND THEREFORE $(*)$ BECOMES:

$$\int_{\mathbb{R}^N} \underbrace{U^2 + V^2}_{\geq 0} = 0$$

AND THEREFORE $\underbrace{U^2}_{\geq 0} + \underbrace{V^2}_{\geq 0} = 0$

HENCE $U = 0$ AND $V = 0$ IN $\mathbb{R}^N$.