HW 11 – SOLUTIONS

#1 STEP 1: SUPPOSE $U(x_1, x_2) = X_1(x_1) + X_2(x_2)$ FOR SOME FUNCTIONS X_1 & X_2

PLUGGING THIS GUESS INTO THE PDE, WE GET:

$$(X_1+X_2)_{x_1}^2 \, (X_1+X_2)_{x_1x_1} + 2(X_1+X_2)_{x_1}(X_1+X_2)_{x_2}(X_1+X_2)_{x_1x_2}$$

$$+ (X_1+X_2)_{x_2}^2 \, (X_1+X_2)_{x_2x_2} = 0$$

BUT USING $(X_1+X_2)_{x_1} = X_1'$ AND $(X_1+X_2) = X_2'$, WE GET

$$(X_1')^2 X_1'' + (X_2')^2 X_2'' = 0$$

$$\underbrace{(X_1')^2 X_1''}_{\text{ONLY DEPENDS ON } x_1} = \underbrace{-(X_2')^2 X_2''}_{\text{ONLY DEPENDS ON } x_2} = \lambda \leftarrow \text{CONSTANT}$$

STEP 2

$$X_1''(X_1')^2 = \lambda$$

$$\left[\frac{1}{3}(X_1')^3\right]' = \lambda$$

$$\frac{1}{3}(X_1')^3 = \lambda x_1 + C$$

$$\Rightarrow X_1' = (3\lambda x_1 + 3C))^{\frac{1}{3}}$$

$$\Rightarrow X_1(x_1) = \frac{1}{3\lambda}\,\frac{3}{4}\,(3\lambda x_1 + 3C)^{\frac{4}{3}} + C''$$

$$\Rightarrow X_1(x_1) = \frac{1}{4\lambda}(3\lambda x_1 + 3C)^{\frac{4}{3}} + C'$$

STEP 3 $\quad -(X_2')^2 X_2'' = \lambda$

$$\Rightarrow \quad (X_2')^2 X_2'' = -\lambda$$

$$\left[\frac{1}{3}(X_2')^3\right]' = -\lambda$$

$$\frac{1}{3}(X_2')^3 = -\lambda x_2 + C''$$

$$X_2' = (-3\lambda x_2 + 3C'')^{\frac{1}{3}}$$

$$X_2 = -\frac{1}{4\lambda}(-3\lambda x_2 + 3C'')^{\frac{4}{3}}$$

STEP 4 : CONCLUSION

HENCE $\quad U(x_1; x_2) = X_1(x_1) + X_2(x_2)$

$$= \frac{1}{4\lambda}(3\lambda x_1 + 3C)^{\frac{4}{3}} - \frac{1}{4\lambda}(-3\lambda x_2 + 3C'')^{\frac{4}{3}}$$

NOW SINCE WE WANT A NONTRIVIAL SOLUTION, WE CAN LET $C, C', C'', C''' = 0$ AND $\lambda = \frac{1}{3}$, SO

$$U(x_1, x_2) = \frac{3}{4}(x_1)^{\frac{4}{3}} - \frac{3}{4}(-x_2)^{\frac{4}{3}}$$

$$\boxed{U(x_1, x_2) = \frac{3}{4}\cdot\left(x_1^{\frac{4}{3}} - x_2^{\frac{4}{3}}\right)}$$

ADDITIONAL PROBLEM #1

$$\begin{cases} U_{tt} = U_{xx} \\ U(0,t) = U(\pi,t) = 0 \\ U(x,0) = \sin(4x) + 7\sin(5x) \\ U_t(x,0) = 2\sin(2x) + \sin(3x) \end{cases}$$

STEP 1 GUESS $U(x,t) = X(x)T(t)$

PLUG INTO $U_{tt} = U_{xx}$

$$(X(x)T(t))_{tt} = (X(x)T(t))_{xx}$$

$$X(x)T''(t) = X''(x)T(t)$$

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{ONLY DEPENDS ON } x} = \underbrace{\frac{T''(t)}{T(t)}}_{\text{ONLY DEPENDS ON } t} = \lambda$$

HENCE $\begin{cases} X'' = \lambda X \\ T'' = \lambda T \end{cases}$

STEP 2 EQUATION FOR X

NOW $U(0,t) = X(0)\cancel{T(t)} = 0 \Rightarrow X(0) = 0$

$U(\pi,t) = X(\pi)\cancel{T(t)} = 0 \Rightarrow X(\pi) = 0$

HENCE WE NEED TO SOLVE $\begin{cases} X'' = \lambda X \\ X(0) = 0,\ X(\pi) = 0 \end{cases}$

CASE 1 $\lambda > 0$, HENCE $\lambda = \omega^2$ AND $X'' = \lambda X \Rightarrow X'' = \omega^2 X$

THIS HAS SOLUTION $X(x) = Ae^{\omega x} + Be^{-\omega x}$

Now $X(0) = A + B = 0 \Rightarrow B = -A \Rightarrow X(x) = Ae^{\omega x} - Ae^{-\omega x}$

$X(\pi) = Ae^{\omega\pi} - Ae^{-\omega\pi} = 0 \Rightarrow \cancel{A}e^{\omega\pi} = \cancel{A}e^{-\omega\pi}$

$\Rightarrow \omega\pi = -\omega\pi$

$\Rightarrow \omega = 0 \Rightarrow\Leftarrow$

CASE 2 $\lambda = 0$ AND $X'' = \lambda X \Rightarrow X'' = 0$

$\Rightarrow X(x) = A + Bx$

$X(0) = A = 0$, so $X(x) = Bx$

And $X(\pi) = B\pi = 0 \Rightarrow B = 0 \Rightarrow X \equiv 0 \Rightarrow\Leftarrow$

CASE 3 $\lambda < 0$, so $\lambda = -\omega^2$ $(\omega > 0)$ and $X'' = \lambda X \Rightarrow X'' = -\omega^2 X$

$\Rightarrow X(x) = A\cos(\omega x) + B\sin(\omega x)$

$X(0) = A = 0$, so $X(x) = B\sin(\omega x)$

$X(\pi) = B\sin(\pi\omega) = 0 \Rightarrow \sin(\pi\omega) = 0 \Rightarrow \pi\omega = \pi m$ $(m = 1, 2, 3, \dots)$

$\Rightarrow \underline{\omega = m}$ $(m = 1, 2, 3, \dots)$

AND FOR EACH m, $X(x) = \sin(mx)$ IS A SOL

STEP 3 EQUATION FOR T

$$\frac{T''}{T} = \lambda \Rightarrow T'' = \lambda T$$

Now $\forall\, m = 1, 2, \dots$, $\lambda = -\omega^2 = -m^2$, so GET $T'' = -m^2 T$

$\Rightarrow T(t) = A_m \cos(mt) + B_m \sin(mt)$

STEP 4 LINEAR COMBOS

FOUND $\forall M = 1, 2, \ldots$ $X(x) = \sin(Mx)$, $T(t) = A_M \cos(Mt) + B_M \sin(Mt)$

USING $U(x,t) = X(x)\,T(t)$ AND TAKING LINEAR COMBOS, WE FIND THAT

$$U(x,t) = \sum_{M=1}^{\infty} \sin(Mx)\left(A_M \cos(Mt) + B_M \sin(Mt)\right)$$ IS A SOLUTION

STEP 5 INITIAL CONDITIONS

$$U(x,0) = \sum_{M=1}^{\infty} A_M \sin(Mx) = \sin(4x) + 7\sin(5x)$$

WHICH TELLS YOU THAT $\underline{A_4 = 1}$, $\underline{A_5 = 7}$, ALL THE OTHER $A_M = 0$

$$U_t(x,0) = \sum_{M=1}^{\infty} M B_M \sin(Mx) = 2\sin(2x) + \sin(3x)$$

WHICH TELLS YOU THAT $2B_2 = 2 \Rightarrow \underline{B_2 = 1}$, $3B_3 = 1 \Rightarrow \underline{B_3 = \frac{1}{3}}$
AND ALL THE OTHER $B_M = 0$

STEP 6 CONCLUSION

$$\boxed{\begin{aligned} U(x,t) = {} & \sin(2x)\sin(2t) + \tfrac{1}{3}\sin(3x)\sin(3t) \\ & + \sin(4x)\cos(4t) + 7\sin(5x)\cos(5t) \end{aligned}}$$

ADDITIONAL PROBLEM #2

(a)
$$\int_{B(0,1)} |U(x)|^2\, dx = \int_{B(0,1)} \frac{1}{|x|^{2p}}\, dx$$

$$= \int_0^1 \int_{\partial B(0,r)} \frac{1}{r^{2p}}\, dS(y)\, dr$$

$$= \int_0^1 \frac{1}{r^{2p}}\, N\alpha(N)\, r^{N-1}\, dr$$

$$= N\alpha(N) \int_0^1 r^{(N-1-2p)}\, dr$$

$$= N\alpha(N) \times \begin{cases} \left[\dfrac{r^{N-2p}}{N-2p}\right]_{r=0}^{r=1} & \text{IF } N \neq 2p \\ \left[\ln(r)\right]_0^1 & \text{IF } N = 2p \end{cases}$$

NOTICE THAT THIS INTEGRAL IS FINITE IFF $N-2p>0$, THAT IS $p<\frac{N}{2}$

HENCE $U \in L^2(B)$ IFF $\boxed{p<\frac{N}{2}}$

(b)
$$U(x) = \frac{1}{|x|^p} = \frac{1}{\left((x_1)^2 + \quad + (x_N)^2\right)^{\frac{p}{2}}} = \left((x_1)^2 + \quad + (x_N)^2\right)^{-\frac{p}{2}}$$

HENCE
$$U_{x_i} = \left(-\frac{p}{2}\right)\left((x_1)^2 + \quad + (x_N)^2\right)^{-\frac{p}{2}-1}(\cancel{2}x_i)$$

$$= \frac{-p\, x_i}{\left((x_1)^2 + \cdots + (x_N)^2\right)^{\frac{p}{2}+1}} = \frac{-p\, x_i}{\left((x_1)^2 + \cdots + (x_N)^2\right)^{\frac{p+2}{2}}} = \frac{-p\, x_i}{|x|^{p+2}}$$

THEREFORE, OUR GUESS IS THAT

$$DU(x) = -p\frac{x}{|x|^{p+2}}$$

(c) LET $\varphi \in C_c^\infty(B)$ BE ARBITRARY AND $i = 1, \dots, N$

LET $\varepsilon > 0$ BE GIVEN, THEN

$$\int_B u\,\varphi_{x_i} = \underbrace{\int_{B(0,\varepsilon)} u\,\varphi_{x_i}}_{A} + \underbrace{\int_{B \setminus B(0,\varepsilon)} u\,\varphi_{x_i}}_{B}$$

NOW $|A| \leq \int_{B(0,\varepsilon)} \frac{1}{|x|^p} \underbrace{|\varphi_{x_i}|}_{\leq C}$ SINCE $\varphi \in C_c^\infty(B)$

$$\leq C \int_{B(0,\varepsilon)} \frac{1}{|x|^p}$$

$$= C \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{r^p}\, dS(y)\, dr$$

$$= C \int_0^\varepsilon \frac{1}{r^p} N\alpha(N) r^{N-1}\, dr$$

$$= C \int_0^\varepsilon r^{N-1-p}\, dr$$

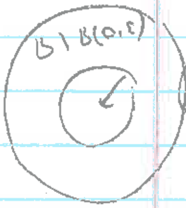
NOTICE $N-1-p > N-1-\frac{N}{2} = \frac{N}{2} - 1 > -1$ ($p < \frac{N}{2}$)

$$= C\left[\frac{r^{N-p}}{N-p}\right]_0^\varepsilon$$

$N-p > N-\frac{N}{2} = \frac{N}{2} > 0$

$$= C\left(\frac{\varepsilon^{N-p}}{N-p} - 0\right) = C\varepsilon^{N-p} \longrightarrow 0 \text{ AS } \varepsilon \to 0$$

ON THE OTHER HAND, FOR (B), INTEGRATING BY PARTS AND USING $\varphi \in C_c^\infty(B)$, WE GET



$$(B) = \int_{B\setminus B(0,\varepsilon)} U\varphi_{x_i} = \int_{\partial B(0,\varepsilon)} U\varphi\,\nu^i - \int_{B\setminus B(0,\varepsilon)} (U_{x_i})\,\varphi$$

$$= \underbrace{\int_{\partial B(0,\varepsilon)} U\varphi \underbrace{\left(\frac{-x_i}{|x|}\right)}_{\nu \text{ POINTS INWARDS}}}_{(C)} - \underbrace{\int_{B\setminus B(0,\varepsilon)} \underbrace{\left(-p\,\frac{x_i}{|x|^{p+2}}\right)}_{U \text{ IS SMOOTH } \underline{\text{OUTSIDE}} \text{ OF } B(0,\varepsilon)}\varphi}_{(D)}$$

NOW

$$|(C)| \le \int_{\partial B(0,\varepsilon)} U \underbrace{|\varphi|}_{\le C} \underbrace{\left|\frac{-x_i}{|x|}\right|}_{\le 1}$$

$$\le C \int_{\partial B(0,\varepsilon)} \frac{1}{|x|^p}\, dS(x)$$

$$= C\,\frac{1}{\varepsilon^p}\, N\alpha(N)\,\varepsilon^{N-1}$$

$$= CN\alpha(N)\,\varepsilon^{N-1-p} \longrightarrow 0 \quad \underline{\text{IFF}} \quad N-1-p > 0, \text{ THAT IS } \underline{p < N-1}$$

NOW IN ORDER TO APPLY THE DOMINATED CONVERGENCE THEOREM TO (D), WE NEED TO MAKE SURE THAT

$$\int_B \left|-p\,\frac{x_i}{|x|^{p+2}}\right| |\varphi| < \infty$$, WHICH WOULD BE ENOUGH IF

$$\int_B \frac{1}{|x|^{p+1}} < \infty,$$ BECAUSE THEN

$$\int_B \left| -p \frac{x_i}{|x|^{p+2}} \right| \underbrace{|\varphi|}_{\leq C} \leq \int_B p \frac{\overset{\leq |x|}{|x_i|}}{|x|^{p+2}} = p \int_B \frac{1}{|x|^{p+1}} < \infty$$

BUT $$\int_B \frac{1}{|x|^{p+1}} = \int_0^1 \int_{\partial B(0,r)} \frac{1}{r^{p+1}} \, dS(y) \, dr$$

$$= \int_0^1 \frac{1}{r^{p+1}} N\alpha(N) r^{N-1} \, dr$$

$$= \int_0^1 N\alpha(N) \, r^{N-p-2} \, dr$$

$$= N\alpha(N) \left[\frac{r^{N-p-1}}{N-p-1} \right]_0^1$$

NOTICE THAT IF $p < N-1$, THEN $N-p-1 > 0$

$$= \frac{N\alpha(N)}{N-p-1} < \infty$$

HENCE, BY THE DCT APPLIED TO (D), IF $\varepsilon \to 0$, (D) GOES TO

$$-\int_B \left(-\frac{p x_i}{|x|^{p+2}} \right) \varphi$$

THEREFORE LETTING $\varepsilon \to 0$ IN (A) + (B), WE ULTIMATELY GET

$$\int_B u \varphi_{x_i} = -\int_B \left(-\frac{p x_i}{|x|^{p+2}} \right) \varphi$$ AND HENCE $Du = -\frac{p x}{|x|^{p+2}}$

(d) Lastly,

$$\int_B |DU|^2 = \int_B p^2 \frac{|x|^2}{|x|^{2p+4}} \, dx$$

$$= p^2 \int_B \frac{1}{|x|^{2p+2}} \, dx$$

$$= p^2 \int_0^1 \int_{\partial B(0,r)} \frac{1}{r^{2p+2}} \, dS(y) \, dr$$

$$= p^2 \int_0^1 \frac{1}{r^{2p+2}} N\alpha(N) r^{N-1}$$

$$= C \int_0^1 r^{N-1-2p-2} = C \int_0^1 r^{N-2p-3}$$

$$= C \begin{cases} \left[\dfrac{r^{N-2p-2}}{N-2p-2} \right]_0^1 & \text{IF } N \neq 2p+2 \\ \left[\ln(r) \right]_0^1 & \text{IF } N = 2p+2 \end{cases}$$

$< \infty$ IFF $N > 2p+2$, THAT IS $p < \frac{N}{2} - 1$

CONCLUSION THEREFORE, WE FOUND THAT IN ORDER TO GUARANTEE $U \in H^1(B)$, WE NEED

1) $p < \frac{N}{2}$
2) $p < N-1$
3) $p < \frac{N}{2} - 1$

NOTICE 3) $\Longrightarrow$ 1) & 2), HENCE WE MUST HAVE $p < \frac{N}{2} - 1$

THEREFORE $U \in H^1(B)$ IFF $\boxed{p < \frac{N}{2} - 1}$