Math 453 – Homework 11 (the last one!)

Peyam Tabrizian

Friday, May 12, 2017

This assignment is due on **Friday, May 12, at 11:50 AM**

Reading: Sections 4.1, 5.1, 5.2, (6.1, 6.2), 8.1.1 - 8.1.2

Note: Chapter 6, although covered in lecture on 05/12, will **not** be on the final. As far as Chapter 8 is concerned, there are only two things I could ask you on the final: either derive the Euler-Lagrange equation in general (as in 8.1.1), or derive it in specific cases (like the Additional Problem on HW 5).

Chapter 4: 1, AP1

Chapter 5: AP2

Hint for 1: Try $u(x_1, x_2) = X_1(x_1) + X_2(x_2)$.

Additional Problem 1: Mimic the derivation presented in lecture to solve the following boundary-value problem for the wave equation for $n = 1$ using separation of variables:

$$\begin{cases} u_{tt} = u_{xx} & 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi, t) = 0 & t > 0 \\ u(x,0) = \sin(4x) + 7\sin(5x) & 0 < x < \pi \\ u_t(x,0) = 2\sin(2x) + \sin(3x) & 0 < x < \pi \end{cases}$$

(TURN PAGE for Additional Problem 2)

Additional Problem 2: In this problem, our goal is to figure out for which $p > 0$, the following function is in $H^1(B)$, where $B = B(0,1)$ is the open unit ball in $\mathbb{R}^n$:

$$u(x) = \frac{1}{|x|^p}$$

Note: If you're *completely* stuck, look at Example 3 on page 260.

(a) First of all, using the onion formula, figure out for which p we have $u \in L^2(B)$.

(b) Formally (= without worrying about any singularities), find a guess for Du.

Hint: If you're not comfortable differentiating absolute values, it might be useful to write $|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ and to calculate u_{x_i} for $i = 1, \cdots, n$.

(c) Using **the definition** of the weak derivative (either the one in lecture or the one on page 256), show that in fact the formula you gave in (b) is the weak derivative of u.

Hint: Basically, the formula you gave in (b) works, except that Du has a singularity at 0, which might cause problems. We'd like to 'isolate' this singularity and show that it's not too bad. So given $\epsilon > 0$, separate your region B into two pieces: $B(0,\epsilon)$ and $B \backslash B(0,\epsilon)$. On the first region, show by estimating that your terms go to zero as ϵ goes to 0. On the second region, integrate by parts and use the fact that because we're away from 0, you can legitimately differentiate u using your formula in (b). Beware that ϕ is not zero on $\partial B(0,\epsilon)$, so you'll also have to estimate those boundary terms and show that they go to 0. Conclude.

(d) For which p is the Du you found in (b) in $L^2(B)$? Hence, for which p is u in $H^1(B)$?