HW 1 - SOLUTIONS

#5 LET $g(t) = f(tx)$

APPLYING TAYLOR'S THEOREM IN ONE VARIABLE, WE HAVE

$$(*) \qquad g(t) = \sum_{M=0}^{K} \frac{1}{M!} g^{(M)}(0)\, t^M + g^{(K+1)}(\xi) \frac{t^{K+1}}{(K+1)!}$$

FOR SOME $\xi \in (0,t)$

NOW SINCE $g(t) = f(tx) = f(tx_1, \ldots, tx_N)$,

WE HAVE

$$g'(t) = \sum_{i=1}^{N} x_i\, f_{x_i}(tx_1, \ldots, tx_N) \qquad \text{(BY THE CHAIN RULE)}$$

$$g''(t) = \sum_{i=1}^{N} x_i \sum_{j=1}^{N} x_j\, f_{x_i x_j}(tx_1, \ldots, tx_N)$$

$$= \sum_{i,j=1}^{N} x_i x_j\, f_{x_i x_j}(tx_1, \ldots, tx_N)$$

AND INDUCTIVELY (YOU DON'T HAVE TO DO THE INDUCTION),

$$g^{(M)}(t) = \sum_{i_1, \ldots, i_M = 1}^{N} x_{i_1} \cdots x_{i_M}\, f_{x_{i_1} \cdots x_{i_M}}(tx_1, \ldots, tx_N)$$

WE WANT TO REWRITE THIS SUM IN MULTI-INDEX NOTATION

LET α BE THE MULTI-INDEX CORRESPONDING TO A GIVEN $(i_1, \dots, i_M)$

FOR EXAMPLE, IF $(i_1, \dots, i_M) = (1,2,1,1,2,3,1)$

$\alpha = (4,2,1)$ (FOUR ONES, TWO TWOS, ONE THREE)

NOTICE BY THE HINT AND BY EQUALITY OF MIXED PARTIAL DERIVATIVES GIVEN A MULTI-INDEX α, THERE ARE EXACTLY $\frac{|\alpha|!}{\alpha!}$ PARTIAL DERIVATIVES THAT ARE EQUAL TO $D^\alpha f$.

THEREFORE, GROUPING THE TERMS CORRESPONDING TO THE SAME MULTI-INDEX TOGETHER, WE GET THAT THE ABOVE SUM (IN MULTI-INDEX NOTATION) BECOMES

$$g^{(M)}(t) = \sum_{|\alpha|=M} \frac{|\alpha|!}{\alpha!} x^\alpha D^\alpha f(tx_1, \dots, tx_N)$$

IN PARTICULAR,

$$g^{(M)}(0) = \sum_{|\alpha|=M} \frac{|\alpha|!}{\alpha!} x^\alpha D^\alpha f(0) = \sum_{|\alpha|=M} \frac{M!}{\alpha!} x^\alpha D^\alpha f(0)$$

AND SO, $(*)$ BECOMES

$$f(tx) = g(t) = \sum_{M=0}^{K} \frac{1}{\cancel{M!}} \sum_{|\alpha|=M} \frac{\cancel{M!}}{\alpha!} x^\alpha D^\alpha f(0)\, t^M$$

$$+ \sum_{|\alpha|=K+1} \frac{\cancel{(K+1)!}}{\alpha!} x^\alpha D^\alpha f(\beta x) \frac{t^{K+1}}{\cancel{(K+1)!}}$$

$$= \sum_{M=0}^{K} \sum_{|\alpha|=M} \frac{1}{\alpha!} D^\alpha f(0)\, x^\alpha t^M + \sum_{|\alpha|=K+1} x^\alpha D^\alpha f(\beta x)\, t^{K+1}$$

$$f(tx) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0)\, x^\alpha t^{n} + \sum_{|\alpha| = k+1} x^\alpha D^\alpha f(\xi x)\, t^{k+1}$$

NOW LETTING $t = 1$, WE GET

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0)\, x^\alpha + \sum_{|\alpha| = k+1} x^\alpha D^\alpha f(\xi x)$$

NOW, JUST AS A TECHNICALITY, WE HAVE:

$$\left| \sum_{|\alpha| = k+1} x^\alpha D^\alpha f(\xi x) \right| \leq \sum_{|\alpha| = k+1} |x^\alpha| \underbrace{|D^\alpha f(\xi x)|}_{\leq C, \text{ SINCE } f \text{ IS SMOOTH}}$$

$$\leq C \sum_{|\alpha| = k+1} |x^\alpha|$$

NOW GIVEN $x = (x_1, \ldots, x_n)$ AND $\alpha = (\alpha_1, \ldots, \alpha_n)$, WE GET

$$|x^\alpha| = |x_1^{\alpha_1} \cdots x_n^{\alpha_n}| = |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n}$$

WE $|x_i| \leq \sqrt{\sum_{i=1}^{n} |x_i|^2} = |x|$

$$\leq |x|^{\alpha_1} \cdots |x|^{\alpha_n}$$

$$= |x|^{\alpha_1 + \cdots + \alpha_n} = |x|^{|\alpha|}$$

DOES NOT DEPEND ON α

AND SO, IN THE ABOVE, WE GET

$$\left| \sum_{|\alpha| = k+1} x^\alpha D^\alpha f(\xi x) \right| \leq C \sum_{|\alpha| = k+1} |x|^{|\alpha|} = C \sum_{|\alpha| = k+1} |x|^{k+1}$$

$$= C |x|^{k+1} \left(\sum_{|\alpha| = k+1} 1 \right) \leq C \quad \leq C |x|^{k+1}$$

AND THEREFORE $\sum_{|\alpha|=K+1} x^\alpha D^\alpha f(?) = O(|x|^{K+1})$ BY DEFINITION, AND SO WE GET:

$$\boxed{f(x) = \sum_{|\alpha|\leq K} \frac{1}{\alpha!} D^\alpha f(0)\, x^\alpha + O(|x|^{K+1})}$$ TA-DAAA!

#2 $v(x) = u(Ox)$

GIVEN $i = 1, \dots, N$,

$$V_{x_i} = \sum_{j=1}^{N} U_{x_j}(Ox)\,((Ox)_j)_{x_i}$$

WHERE $(Ox)_j = \sum_{K=1} O_{jK} x_K$ (THE j^{TH} ENTRY OF Ox)

IN PARTICULAR, $((Ox)_j)_{x_i} = O_{ji}$

AND THEREFORE

$$V_{x_i} = \sum_{j=1}^{N} U_{x_j}(Ox)\, O_{ji}$$

NOW, USING THE CHAIN RULE AGAIN, WE HAVE:

$$V_{x_i x_i} = \sum_{K=1}^{N} \sum_{j=1}^{N} U_{x_j x_K}(Ox)\,((Ox)_K)_{x_i}\, O_{ji}$$

NOW $(Ox)_K = \sum_{l=1}^{N} O_{Kl}\, x_l$, SO

$\left((Ox)_k \right)_{x_i} = O_{ki}$ AND THEREFORE

$$V_{x_i x_i} = \sum_{k=1}^{N} \sum_{j=1}^{N} U_{x_j x_k}(Ox) \; O_{ki} O_{ji}$$

THEREFORE :

$$\Delta V = \sum_{i=1}^{N} V_{x_i x_i}$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{N} \sum_{j=1}^{N} U_{x_j x_k}(Ox) \; O_{ki} O_{ji}$$

$$= \sum_{k=1}^{N} \sum_{j=1}^{N} U_{x_j x_k}(Ox) \underbrace{\sum_{i=1}^{N} O_{ki} O_{ji}}_{= 1 \text{ IF } k=j, \; 0 \text{ IF } k \neq j}$$

SINCE O IS ORTHOGONAL

$$= \sum_{k=1}^{N} U_{x_k x_k}(Ox)$$

$$= \Delta U (Ox)$$

$= 0$ SINCE $\Delta U = 0$

AND HENCE $\Delta V = 0$ AS WELL !