MATH 453 – FINAL EXAM – STUDY GUIDE

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The Math 453 – Final Exam can be picked up from Hopkins Hall from Saturday, May 13, up to and including Sunday, May 21, and you have **24** hours to take it. It covers *everything* that we've learned in this course, including is sections 2.1 – 2.4, 3.2, 4.1.1, 5.2, 5.6.1, and 8.1.2. It will be a **closed** book and closed notes-exam, and no cheat-sheets will be allowed. Note that I will be out of town on May 19 – 21, and might be unable to answer questions during that time.

This is a study guide for the exam, and (hopefully) covers *everything* you need to know for the exam. For your convenience, I have merged this with the midterm study guide.

Note: Note that, unless otherwise indicated, you do **NOT** have to memorize any formulas, such as the fundamental solutions for Laplace's equation and for the heat equation, or the characteristic ODE. I will provide them to you if necessary

Main concepts: Laplace's equation (section 2.2), Heat equation (section 2.3), Wave equation (section 2.4), Method of characteristics (section 3.2), Separation of variables (section 4.1.1), Sobolev spaces (section 5.2), Calculus of Variations (section 8.1.2)

THEOREMS WHOSE STATEMENTS AND PROOFS YOU NEED TO KNOW (= MEMORIZE!!!)

Note: Don't worry too much about the smoothness assumptions (about continuity etc.), I'm more interested in the formulas/results. The way I stated and proved the results in lecture is completely fine, and you don't have to be as precise as the book. But of course, you have to say "Suppose $\Delta u = 0$ or suppose $u_t - \Delta u = 0$"

- **The mean-value formulas for Laplace's equation** (both versions)
- The converse to the mean-value property (Theorem 3 on page 26)
- **The strong maximum principle for Laplace's equation**

Date: Tuesday, May 2, 2017.

- Using the strong maximum principle to show positivity (page 27)
- **Uniqueness of solutions of Poisson's equation using the maximum principle** (Theorem 5 on page 28)
- The decay estimates (Theorem 7 on page 29), but *only* in the case of $|u(x)|$ (see lecture)
- Liouville's theorem (Theorem 8 on page 30)
- **Uniqueness of solutions of Poisson's equation using energy methods** (Theorem 16 on page 42)
- **Dirichlet's principle** (Theorem 17 on pages 42 and 43; also see Chapter 8)
- How to show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, and hence how to show that the integral of the fundamental solution is 1 (bottom of page 46).
- Infinite propagation speed of heat equation (page 48)
- Solution of homogeneous problem of the heat equation with general initial data (bottom of page 51)
- **The strong maximum principle for the Heat Equation** (Theorem 4 on page 55; I would provide you with the mean value formula for the heat equation if necessary)
- Infinite Propagation speed again (page 57)
- Uniqueness on bounded domains and Uniqueness of the Cauchy problem (Theorem 5 on page 57 and Theorem 7 on page 59)
- **Uniqueness of heat equation using energy methods**
- **Solution of the transport equation** (pages 18 and 19)
- **D'Alembert's formula** (pages 67-68). I would provide you with the solution of the inhomogeneous transport equation if necessary. You may also use the method in Problem 19 in HW 8.
- Heat and Wave equations on the half-line (Page 69 and also the Additional Problem in HW 6 and the Additional Problem 1 on HW 9)
- Euler-Poisson-Darboux equation (Lemma 1 on page 70): I would provide you with the statement if necessary, but you'll need to show me how to obtain $(r^{n-1}U_r)_r = r^{n-1}U_{tt}$, and don't worry about part 1. of the proof.
- Kirchoff's formula (pages 71-72) I would provide you with the transformation and the Euler-Darboux equation if necessary. All you'll need to understand is that the transformation turns $\widetilde{U}$ into a solution of the wave equation, and then you can use D'Alembert.
- **Uniqueness of solutions for the wave equation using energy methods** (Theorem 5 on pages 82-83)

- Finite Propagation speed for the wave equation (Theorem 6 on page 84); I'd provide you with the definition of $K(x_0, t_0)$, but you'll have to come up with the energy yourself!
- Know how to show that the characteristics for conservation laws are straight lines (Example 5 on page 111)
- Know how to derive Hamilton's ODEs (Example 6 on page 113)
- Uniqueness of weak derivatives (page 257)
- **Derivation of the Euler-Lagrange equations** (pages 454-456)

THEOREMS WHOSE STATEMENTS (BUT **NOT** PROOFS) YOU NEED TO KNOW (= MEMORIZE)

Note: Of course you have to know the statements of the theorems in the above sections as well!

- Divergence Theorem
- Dominated Convergence Theorem
- Integration by parts (Theorems 1, 2, 3 on pages 711 and 712)
- Polar Coordinates formula (Theorem 4 on page 712)
- Solving Poisson's equation (Theorem 1 on page 23)
- Harnack's inequality (Theorem 11 on page 32)
- The decay estimates (Theorem 7 on page 29), but only in the case of the first derivative (see lecture)
- Smoothness (Theorem 6 on page 28)
- The true second-derivative test (see handout)
- Properties of mollifiers (Theorem 7(i) and 7(ii) on page 714, don't worry about what U_ϵ is)
- Cauchy-Schwarz and Cauchy's inequalities ((a) and (i) on pages 706 and 708)
- Formula of the solution of the initial-value problem of the heat equation (Theorem 1 on page 47)
- Formula of the solution of the nonhomogeneous problem of the heat equation (Theorem 2 on page 50)
- Maximum principle for the Cauchy problem (Theorem 6 on page 57)
- Regularity of solutions to the heat equation (Theorem 8 on page 59)
- Backwards uniqueness of heat equation (Theorem 11 on page 64)
- Duhamel's principle applied to the wave equation (bottom of page 80)
- Definition of the weak derivative
- Definition of $L^2(W)$, $H^1(W)$, $\|u\|_{H^1(W)}$, H_0^1
- Poincaré inequality: For any $u \in H_0^1(W)$,

$$\|u\|_{L^2(W)} \leq C \|Du\|_{L^2(W)}$$

Chapter 1: Introduction

- This chapter is just an overview of PDEs, so don't worry about it
- Know what $D^\alpha u$ means, where α is a multi-index
- I will **NOT** ask you about Problem 5 in Chapter 1 (which was on HW 1).

Chapter 2: Laplace's equation

Introduction.

- Skip the *physical* derivation of Laplace's equation (in terms of net flux)
- Know the fact that if $\int_V f dx = 0$ for all regions V, then $f \equiv 0$
- Know the divergence theorem

Section 2.2.1: Fundamental Solution.

- You **DON'T** need to know how to show that solutions to Laplace's equation are invariant under rotation (Problem 2 in Chapter 2) and you **DON'T** need to know how to derive the fundamental solution of Laplace's equation.
- Know the definition of convolution.
- Know the dominated convergence theorem, the polar coordinates formula, and the integration by parts formula, and how to apply them. The examples covered in Lecture 4, on HW 2, and Problems 1 and 2 in HW3 give you great practice with those concepts.
- Know the definition of the normal derivative, and in particular the formula for ν on $\partial B(0,r)$ and $\partial B(x,r)$.
- Know the formula for the solution of Poisson's equation, but you do **NOT** need to know how to derive it.
- Know how to show that if f is continuous, then the average integral of f on $\partial B(x,r)$ goes to $f(x)$ (see HW 3)

Section 2.2.2: Mean-Value Formulas.

- Be comfortable with change of variables! Notice how much we've used this throughout the course!
- Know the statement and the proof of the mean-value formulas for Laplace's equation
- Don't worry about Problem 3 in Chapter 2 (the mean-value formula with f and g) it's awkward to ask that on an exam

Section 2.2.3: Properties of harmonic functions.

- Know the statement and the proof of the strong (and weak) maximum principle. Also know how to prove it the way they do in Problem 4 of Chapter 2.
- Know how to apply the strong maximum principle to show that a solution u (with $g \geq 0$) is positive (see page 27)
- Know how to apply the strong maximum principle to show that solutions of Poisson's equation on a bounded open connected domain are unique (Theorem 5 on page 28)
- Know the statement of Harnack's inequality (Theorem 11 on page 32). Don't memorize the proof, but notice how the mean-value property is used here!
- For the decay estimates, only know the statement and proof for $|u(x)|$ itself (just like I did in lecture). For the first derivative $|Du(x)|$, know the statement, but not the proof.
- Know Liouville's theorem (Theorem 8 on page 30) and the Representation formula (Theorem 9 on page 30); in that proof you do **not** need to show that v ($\widetilde{u}(x)$ in the book) is bounded.
- Know the statement about smoothness (Theorem 6 on page 28), but do **not** know the proof. You do **not** need to know the definition of $\eta(x)$ or $\eta_\epsilon(x)$, but you **do** need to know what f_ϵ is and the two properties it satisfies (that it's smooth and that it converges to f)

Section 2.2.5: Energy methods.

- Know how to show how to prove uniqueness of Poisson's equation using energy methods.
- Problem 3 on HW 2 provides more practice with energy methods!
- Know the Cauchy-Schwarz inequality and Cauchy's inequalities
- Know how to derive Dirichlet's principle; I may also ask you about more exotic cases like the additional problem on HW 5; also see Chapter 8.
- Know how to show that if $\int_W fg = 0$ for all smooth g, then $f \equiv 0$.

THE HEAT EQUATION

Introduction.

- Again, don't worry about the physical derivation of the heat equation

Section 2.3.1: Fundamental Solution.

- Know the formula, but not the proof, for the solution of the initial-value problem of the heat equation (Theorem 1 on page 47).

- Know how to show that the solution of the heat equation has infinite propagation speed.
- Know the formula, but not the proof, of the solution of the nonhomogeneous problem. That said, again, some of the techniques are very classical (such as integration by parts).
- Don't worry about Problem 13 in Chapter 2
- I could ask you to do something like Problem 14 in Chapter 2, where you transform an equation into the heat equation (see also Problem 1 in Chapter 2).

Section 2.3.2: Mean-value formula.

- **I will provide you with the definitions of parabolic cylinder and parabolic boundary, so don't worry about them.**
- Don't worry about the definition of the heat ball, I will provide it to you if necessary
- Look at the statement of the mean-value formula for the heat equation (Theorem 3 on page 53), but *don't* memorize it. I'd provide you with the statement if necessary. You can also skip the proof, because it is a bit ridiculous!

Section 2.3.3: Properties of solutions.

- Know the statement and the proof of the strong maximum principle for the heat equation, but don't worry too much about the second part of the proof about line segments.
- Know how to derive the infinite propagation speed of the heat equation using the maximum principle (top of page 57)
- Know how to derive uniqueness of the heat equation on bounded domains (Theorem 5 on page 57); I would like to have a little bit more details than in the book, just the way I did in lecture
- Don't worry about the proof of the maximum principle for the Cauchy problem, but know the statement and how to apply it.
- Know how to derive the uniqueness of the Cauchy problem for the heat equation. Note in particular that it says that if there are two **bounded** solutions to the heat equation ($a = 0$), then they are equal.
- **Don't worry** about the proof of regularity of solutions to the heat equation (it's kinda crazy); skip the section on local estimates for solutions to the heat equation
- Remember Problems 16 and 17 on HW 6, and know how to solve the heat equation on the half-line (the Additional Problem in HW 6).

Section 2.3.4: Energy methods.

- **Know the statement and proof of uniqueness using energy methods.** I will **not** provide you with the energy, you should come up with it on your own.
- Know the three equivalent characterizations of convexity.
- You do **not** need to memorize the proof of the backwards uniqueness, but the main parts you need to understand is how to calculate $E''(t)$ and how to get $(E'(t))^2 \leq E(t)E''(t)$. Don't worry about Step 4 in the proof (Step 3 in the book) where I used some analysis.
- For more practice using energy methods, look at the Additional Problem on HW 7.

THE TRANSPORT EQUATION

- **Know how to derive the solution of the transport equation** (both the homogeneous case and the inhomogeneous case). You may use either the method in the book, or the method of characteristics (Problem 4 in HW 10).
- Know how to reduce a PDE to the transport equation (Problem 1 in HW 7)

THE WAVE EQUATION

Introduction.

- Don't worry about the physical derivation of the wave equation, and don't worry about Stokes' Rule (Problem 18 on HW 7)

Section 2.4.1: Solutions by Spherical means.

- **Know how to derive D'Alembert's formula** for the solution of the wave equation for $n = 1$. I would provide you with the solution of the transport equation if necessary. If you wish, you may use the method in Problem 19 on HW 8 instead.
- Know how to solve the wave equation on the half-line (page 69), as well as the variation using evenification (Additional Problem 1 on HW 9).
- Ignore Problems 21 and 22 on HW 8; they are too inappropriate to ask on an exam. Also ignore the problem with the Laplace transform on HW 9.
- You do not need to memorize the statement of the Euler-Poisson-Darboux equation, but know how to derive that $(r^{n-1}U_r)_r = U_{tt}$ (on page 71)
- Know how to derive Kirchoff's formula; I would provide you with the transformation and the Euler-Poisson-Darboux equation if necessary. All you'll need to understand is that the transformation

turns $\widetilde{U}$ into a solution of the wave equation, and then you can use D'Alembert.
- You can skip the derivation of Poisson's formula (again, awkward for an exam)

Section 2.4.2: Nonhomogeneous problem.

- You don't need to memorize the formula for the solution of the inhomogeneous problem for the wave equation, **but** please understand Duhamel's principle and in particular *how* to obtain the solution of the nonhomogeneous problem from the solution of the initial-value problem. I may as well ask you something like: Here's the solution of the initial-value problem of a PDE; find a solution of the inhomogeneous PDE.
- Know the chain rule for integrals:

$$\frac{d}{dt}\int_0^t g(x,t,s)ds = \left(\int_0^t g_t(x,t,s)ds\right) + g(x,t,t)$$

Section 2.4.3: Energy methods.

- **Know how to derive uniqueness of solutions to the wave equation using energy methods** (Theorem 5 on page 82-83)
- Know how to show finite speed of propagation of solutions of the wave equation (Theorem 6 on page 84). I'd provide you with the definition of $K(x_0, t_0)$, but you'd have to come up with the energy yourself.
- For more practice with energy methods, look at Problem 24 on HW 9.

FIRST-ORDER PDE

You are **NOT** responsible for the theory part of this chapter and you do **NOT** need to memorize the characteristic ODE, I will provide them to you if necessary, but know how to use them! There are two things I could ask you in this section: either derive the solution of the transport equation using the method of characteristics (Problem 4 on HW 10), or solve a PDE using the method of characteristics as in Examples 1, 2, 3 in lecture, the part about Conservation laws and Hamilton-Jacobi equations, as well as Problem 5 + the Additional Problem on HW 10.

SEPARATION OF VARIABLES

Any of the problems I've done in this chapter, like Examples 1, 2, 3 in lecture, and Problem 1 and Additional Problem 1 (the Wave equation) on HW

11 are all fair game for the exam. If I ask you something other than Example 3 (the heat equation) or Additional Problem 1 (the Wave equation), I would provide you with the form of u, but in the latter case, you should know what form to use. Fourier series (if I mention them) will **NOT** be on the exam.

Also remember that you'll need to know the solutions to the ODEs $y'' - a^2y = 0$ and $y'' + a^2y = 0$.

SOBOLEV SPACES

- Know the definition of the weak derivative (page 256)
- Know how to show that weak derivatives are unique (Lemma on page 257)
- **Calculate the weak derivative of a function using the definition, or show that a weak derivative does not exist** (Examples 1 and 2 on pages 257 and 248).
- Know the definition of $L^2(W)$, the Sobolev space $H^1(W)$, the norm $\|u\|_{H^1(W)}$, and the Sobolev space $H_0^1(W)$. I will **NOT** ask you about $W^{k,p}(W)$ and about $H^k(W)$ for $k \geq 2$.
- **Know how to show that a function u is in $H^1(W)$.** Notice this entails three things: 1) Showing that u is in $L^2(W)$, 2) Finding Du and showing it's the weak derivative, and 3) Showing that Du is in $L^2(W)$. Look for example at Example 3 (with $p = 2$) and at the Additional Problem on HW 11.
- **Know the Poincaré inequality**: For any $u \in H_0^1(W)$,

$$\|u\|_{L^2(W)} \leq C \|Du\|_{L^2(W)}$$

- Anything else I mention in this chapter you're **NOT** responsible for!

CALCULUS OF VARIATIONS

The only thing you'll need to know in this section is **how to derive the Euler-Lagrange equation** (pages 454-456). I may also ask you to do that for a specific example, like Examples 1, 3, 4 on pages 456-457 (skip Example 2). See also the Additional Problem in HW 5. Anything else in this chapter you're **NOT** responsible for!

SECOND-ORDER ELLIPTIC EQUATIONS

Although I'll talk about it in lecture, you are **NOT** responsible for anything in this chapter!