SOLUTIONS

MATH 200 – FINAL EXAM

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Name: ____

Instructions: Welcome to your Final Exam! You have 150 minutes (= 2 h 30) to take this exam, for a total of 100 points. **There are lots of questions, so do not spend more than 15 minutes on each question.** Remember that you are not only graded on the correctness of your answer, but also on the clarity and completeness of your proofs. Write in full sentences whenever you can. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. Good luck, and may the odds be in your favor! :)

Honor Code: I promise not to communicate with anyone about the exam unless everyone in my party has taken it, and I will not use any books or notes or cheat sheets or personal electronic devices (**including calculators**).

Signature: ____

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		+
Total		100

Date: Saturday, May 13 — Sunday, May 21, 2017.

1. *(10 points)* Determine whether or whether the following reasoning is valid or not. Feel free to add more columns if that helps you.

$$(p \wedge q) \Rightarrow \sim r$$
$$p \vee (\sim q)$$
$$(\sim q) \Rightarrow p$$
$$\therefore\ p \vee r$$

p	q	r	$(p \wedge q) \Rightarrow \sim r$	$p \vee (\sim q)$	$(\sim q) \Rightarrow p$	$p \vee r$
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	T	F	T	T
F	T	F	T	F	T	F
F	F	T	T	T	F	T
F	F	F	T	T	F	F

NOTE THAT, WHENEVER THE PREMISES ARE TRUE, SO IS THE CONCLUSION, HENCE THE REASONING IS VALID

2. *(10 points)* Prove that there are infinitely many prime numbers.

Note: You are allowed to use **without proof** any facts that you know about integers and/or divisibility.

SUPPOSE THERE ARE ONLY FINITELY MANY PRIME NUMBERS $p_1, \cdots, p_m$

LET $N = (p_1 \cdots p_m) + 1$

THEN $N > p_i \ \forall i = 1, \cdots, m$, HENCE N IS NOT PRIME

(SINCE WE ASSUMED THAT THERE ARE ONLY FINITELY MANY PRIME #s)

HENCE THERE MUST BE A PRIME NUMBER p THAT DIVIDES N, SO $p \mid N$ (1)

BUT SINCE p IS PRIME, $p = p_i$ FOR SOME $i = 1, \cdots, N$

(SINCE $p_1, \cdots, p_m$ ARE ALL THE PRIME NUMBERS)

AND THEREFORE $p = p_i \mid (p_1 \cdots p_m)$ (2)

IN PARTICULAR, BY (1) AND (2) WE GET THAT

$p \mid N - (p_1 \cdots p_m) = (p_1 \cdots p_m) + 1 - (p_1 \cdots p_m) = 1$

SO $p \mid 1$ AND HENCE $p = 1$ (SINCE $p > 0$)

BUT THIS $\Rightarrow\Leftarrow$ THE FACT THAT p IS PRIME ! $\Rightarrow\Leftarrow$

HENCE, THERE ARE ∞ MANY PRIME NUMBERS ∎

3. *(10 points)* Show **using strong induction** that if $n \geq 12$, then n cents can be changed using a combination of 3 cent and 7 cent coins.

STRONG INDUCTION LET P_N BE THE PROP "N¢ CAN BE CHANGED USING 3¢ AND 7¢"

BASE CASE N=12 $12 = 4 \times (3¢)$

N=13 $13 = 6 + 7 = 2 \times (3¢) + 1 \times (7¢)$

N=14 $14 = 2 \times (7¢)$

INDUCTIVE STEP SUPPOSE P_K IS TRUE $\forall K = 12, \ldots, N$

THAT IS "K ¢ CAN BE CHANGED USING 3¢ AND 7¢"

CLAIM P_{N+1} IS TRUE, THAT IS "N+1 ¢ CAN BE CHANGED USING 3¢ AND 7¢"

BUT $N+1 = 3 + (N+1-3)$

$= 3 + (N-2)$ ¢

SINCE P_{N-2} IS TRUE, $N-2$ ¢ $= a \times (3¢) + b \times (7¢)$

$= 3¢ + a \times (3¢) + b \times (7¢)$

$= (a+1) \times 3¢ + b \times 7¢$

HENCE N+1 ¢ CAN BE CHANGED USING 3¢ AND 7¢

HENCE P_{N+1} IS TRUE

HENCE, BY STRONG INDUCTION, P_N IS TRUE $\forall N = 12, \ldots$

THAT IS N¢ CAN BE CHANGED USING 3¢ AND 7¢ COINS

4. *(10 points, 5 points each)* The two parts of this problem are independent.

(a) Show that if A, B, C are sets and $B \subseteq (C - A)$, then $A \cap B = \emptyset$

SUPPOSE THERE IS AN $x \in A \cap B$

THEN $x \in B \subseteq C - A$, SO $x \in C - A = C \cap A^c$

IN PARTICULAR $x \in A^c$ SO $x \notin A$

BUT SINCE $x \in A \cap B$, $x \in A$ $\Rightarrow\Leftarrow$

HENCE $A \cap B = \emptyset$

(ALTERNATIVELY: $B \subseteq C - A \Rightarrow A \cap B \subseteq (C - A) \cap A = (C \cap \underbrace{A^c) \cap A}_{\emptyset} = C \cap \emptyset = \emptyset$)

(b) Give an example of something that is not a set and show that it's not a set.

LET $A = \{ x \mid x \notin A \}$

THEN FOR ANY x, EITHER $x \in A$ OR $x \notin A$

CASE 1 $x \in A$, BUT THEN BY DEF, $x \notin A$ $\Rightarrow\Leftarrow$

CASE 2 $x \notin A$, BUT THEN BY DEF $x \in A$ $\Rightarrow\Leftarrow$

HENCE A IS NOT A SET.

5. *(10 points)* In this problem, you may use **without proof** any facts and properties that you know about countable sets. Moreover, you will need to use the following result:

Fundamental Theorem of Algebra: A polynomial of degree n has at most n zeros (where $n \geq 1$).

A real number x is called **algebraic** if it is the zero of a polynomial of degree n with rational coefficients, that is if there is an integer $n \geq 1$ and rational numbers $a_0, a_1, \cdots, a_n$ with $a_n \neq 0$ such that

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Show that the set of algebraic numbers is countable.

Hint: Write the set of algebraic numbers as the countable union of countable sets.

NOTICE THAT THE SET OF ALGEBRAIC NUMBERS IS JUST :

$$ALG = \bigcup_{N=1}^{\infty} \bigcup_{(a_0,\dots,a_N) \in \mathbb{Q}\times\mathbb{Q}\times\cdots\times\mathbb{Q}\times\mathbb{Q}^*} \{x \mid a_N x^N + \dots + a_0 = 0\}$$

"IF THERE IS AN INT $N \geq 1$" , "IF THERE ARE $a_0, \dots, a_N \in \mathbb{Q}$, $a_N \neq 0$"

NOW FOR FIXED $N \geq 1$ & $a_0, \dots, a_N \in \mathbb{Q}$, $a_N \neq 0$, $\{x \mid a_N x^N + \dots + a_0 = 0\}$ HAS AT MOST N ELEMENTS (BY THE FUND THM OF ALG) HENCE IS COUNTABLE

MOREOVER, $\mathbb{Q} \times \mathbb{Q} \times \dots \times \mathbb{Q}^*$ IS COUNTABLE B/C THE PRODUCT OF COUNTABLE SETS IS COUNTABLE (AND $\mathbb{Q}$ IS COUNTABLE).

HENCE $\bigcup_{(a_0,\dots,a_N) \in \mathbb{Q}\times\cdots\times\mathbb{Q}^*} \{x \mid a_N x^N + \dots + a_0 = 0\}$ IS COUNTABLE AS THE COUNTABLE UNION OF COUNTABLE SETS

AND THEREFORE ALG IS COUNTABLE AS THE COUNTABLE UNION (OVER N) OF THE COUNTABLE SETS ABOVE.

6. *(10 points)* In this problem you may use **without proof** any facts that you know about rational numbers.

Define the following relation $\sim$ on $\mathbb{R}^*$ (= the nonzero real numbers):

$$x \sim y \text{ if and only if } \frac{x}{y} \text{ is rational}$$

(a) *(7 points)* Show that $\sim$ is an equivalence relation on $\mathbb{R}^*$

$x \sim x$? $\frac{x}{x} = 1$ is rational ✓

$x \sim y \Rightarrow y \sim x$? IF $\frac{x}{y}$ is rational, THEN $\frac{y}{x} = \frac{1}{\left(\frac{x}{y}\right)} = \frac{1}{\text{rational}}$, which is rational.

$x \sim y \wedge y \sim z \Rightarrow x \sim z$? IF $\frac{x}{y}$ is rational AND $\frac{y}{z}$ is rational, THEN

$\frac{x}{z} = \frac{x}{y} \times \frac{y}{z}$ is rational
(rat) (rat)

(b) *(3 points)* Find [1] (the equivalence class of 1)

$[1] = \{x \in \mathbb{R}^* \mid x \sim 1\}$

$= \{x \in \mathbb{R}^* \mid \frac{x}{1} \text{ is rational}\}$

$= \{x \in \mathbb{R}^* \mid x \text{ is rational}\}$

$= \mathbb{Q}^*$ (THE NONZERO RATIONAL NUMBERS)

7. *(10 points, 5 points each)* The two parts of this problem are independent of each other, and you do **NOT** have to simplify your answer

(a) Urn 1 contains 10 red balls and 25 green balls, and Urn 2 contains 25 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a crooked coin with probability of 0.4 of landing heads. If the coin lands heads, the Urn 1 is chosen; otherwise the Urn 2 is chosen. Then a ball is picked at random from the chosen urn. If the chosen ball is green, what is the probability that it was picked from the Urn 1?

BY BAYES,

$$P(\text{URN 1} \mid G) = \frac{P(G \mid \text{URN 1})\,P(\text{URN 1})}{P(G \mid \text{URN 1})\,P(\text{URN 1}) + P(G \mid \text{URN 2})\,P(\text{URN 2})}$$

$$= \frac{\left(\frac{25}{35}\right)(0.4)}{\left(\frac{25}{35}\right)(0.4) + \left(\frac{15}{40}\right)(0.6)}$$

(b) Today, Café Peyam serves Apple, Blueberry, Chocolate, and Yam Pies. How many different combinations of 20 pies contain at least 8 Apple Pies and at most 5 Blueberry Pies?

SINCE THE FIRST 8 PIES HAVE TO BE APPLE PIES, WE JUST NEED TO KNOW "HOW MANY DIFFERENT SELECTIONS OF 12 PIES CONTAIN ≤ 5 BLUEBERRY PIE

$$|B_{\le 5}| = \text{TOTAL} - |B_{\ge 6}|$$

$$= \binom{15}{12} - \binom{9}{6}$$

12 X
3 |

6 X
3 |

8. *(10 points)*

(a) *(5 points)* Show that $\sum_{k=0}^{n} k\binom{n}{k}\left(\frac{1}{2}\right)^{n-1} = n$

Hint: First use the binomial theorem to expand out $\left(x+\frac{1}{2}\right)^n$. Then differentiate both sides with respect to x, and finally let $x = \frac{1}{2}$.[1]

BY THE BINOMIAL THEOREM WITH $a = x$, $b = \frac{1}{2}$, WE GET

$$\left(x+\frac{1}{2}\right)^{n} = \sum_{k=0}^{n} \binom{n}{k} x^{k} \left(\frac{1}{2}\right)^{n-k}$$

DIFFERENTIATING (USING (1) AND (2)) BELOW, WE GET

$$n\left(x+\frac{1}{2}\right)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} \left(\frac{1}{2}\right)^{n-k}$$

LETTING $x = \frac{1}{2}$, WE GET

$$n = n\left(\frac{1}{2}+\frac{1}{2}\right)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k \left(\frac{1}{2}\right)^{k-1}\left(\frac{1}{2}\right)^{n-k} = \sum_{k=0}^{n} k\binom{n}{k}\left(\frac{1}{2}\right)^{n-1}$$

(b) *(5 points)* Use your answer in (a) to find the expected number of heads when n (fair) coins are tossed.

THE OUTCOMES ARE $k = 0, 1, 2, \cdots, n$ WITH PROBABILITIES $\binom{n}{k}\left(\frac{1}{2}\right)^{n}$,

HENCE $E = \sum_{k=0}^{n} k\binom{n}{k}\left(\frac{1}{2}\right)^{n}$

$$= \sum_{k=0}^{n} k\binom{n}{k}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} = \frac{1}{2}\underbrace{\sum_{k=0}^{n} k\binom{n}{k}\left(\frac{1}{2}\right)^{n-1}}_{n \text{ (BY (a))}} = \boxed{\frac{n}{2}}$$

[1] In case you don't know what a derivative is, it's an operation denoted by $'$ with the following rules:

(1) For any constants c_k and any functions f_k,

$$\left(\sum_{k=0}^{n} c_k f_k(x)\right)' = \sum_{k=0}^{n} c_k f_k'(x)$$

(2) For any $n \geq 0$ and $k \geq 0$,

$$\left(\left(x+\frac{1}{2}\right)^n\right)' = n\left(x+\frac{1}{2}\right)^{n-1} \qquad (x^k)' = kx^{k-1}$$

9. *(10 points, 2 points each)* Give an example (or show that one does not exist)

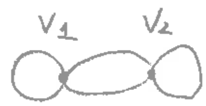
FALSE

(a) A graph with 4 vertices of degrees 1, 1, 1, 4

DOES NOT EXIST, BECAUSE THE TOTAL DEGREE IS $1+1+1+4=7$, BUT WE KNOW THE TOTAL DEGREE MUST BE EVEN.

TRUE

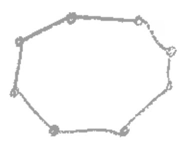
(b) A graph with 2 vertices of total degree 8.



$DEG(G) = DEG(V_1) + DEG(V_2) = 4+4=8$

TRUE

(c) A simple graph with 8 vertices and 8 edges.



FALSE

(d) A tree with 9 edges and 7 vertices.

A TREE WITH 7 VERTICES MUST HAVE $7-1=6$ EDGES

TRUE

(e) A full binary tree of height 3 and total degree [redacted] 28

NOTE: I GAVE THOSE PEOPLE WHO HAD 22 INSTEAD OF 28 (W/O THE CORRECTION) FULL CREDIT (BY DEFAULT)

10. *(10 points, 2 points each)* Give an example (or show that one does not exist)

FALSE

(a) A full binary tree of height 3 and 9 terminal vertices

WE KNOW $t \leq 2^h$, SO $9 \leq 2^3 = 8 \Rightarrow \Leftarrow$

ALTERNATIVELY DEG(G) = 21 BUT DEG(G) MUST BE EVEN.

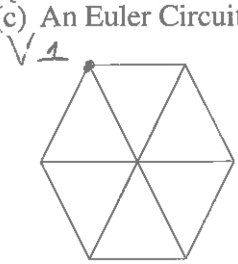
FALSE

(b) A full binary tree with 5 internal vertices and total degree 21.

WE KNOW THAT THIS TREE MUST HAVE $5+1=6$ TERMINAL VERTICES AND HENCE $6+5=11$ TOTAL VERTICES AND HENCE $11-1=10$ TOTAL EDGES AND HENCE DEGREE $= 2 \times$ EDGES $= 2 \times 10 = 20 < 21$.

FALSE

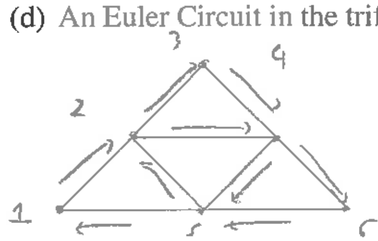
(c) An Euler Circuit in the following graph:



VERTEX V_1 HAS DEGREE 3, WHICH IS ODD (FOR AN EULER CIRCUIT TO EXIST EVERY VERTEX MUST HAVE > 0 AND EVEN DEGREE)

TRUE

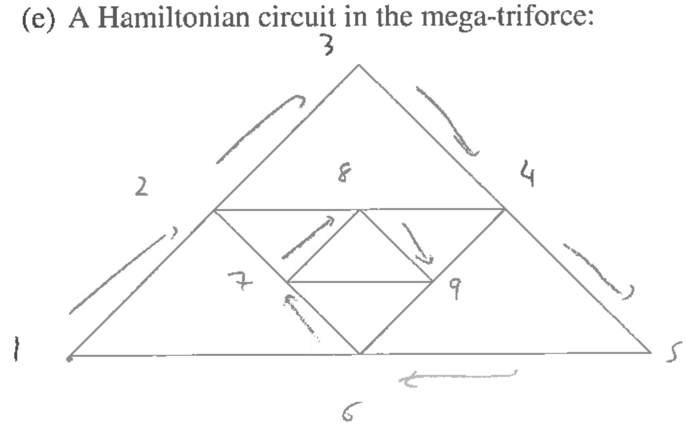
(d) An Euler Circuit in the triforce:



$1 \to 2 \to 3 \to 4 \to 5 \to 2 \to 4 \to 6 \to 5 \to 1$

TRUE

(e) A Hamiltonian circuit in the mega-triforce:



$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9$

11. *(0 points)* **Note:** This problem is worth 0 points and is just meant for the people who are trying to get an A+ in this course. Please thoroughly review your answers to the other questions before tackling this. Also this is not a guarantee for an A+, it all depends on my own discretion.

If A is set, show that there is a one-to-one function from A to $\mathcal{P}(A)$ but no onto function from A to $\mathcal{P}(A)$ (so, in some sense, $\mathcal{P}(A)$ is strictly larger in size than A).

Hint: Give an argument along the lines of Russel's Paradox.

FIRST OF ALL THE FUNCTION $f : A \to P(A)$ GIVEN BY $f(x) = \{x\}$ IS 1-1 B/C $f(x) = f(y) \Rightarrow \{x\} = \{y\} \Rightarrow x = y$

NOW SUPPOSE THAT THERE IS AN ONTO FUNCTION $f : A \to P(A)$

DEFINE $B = \{ x \in A \mid x \notin f(x) \}$

THEN $B \in P(A)$ SO SINCE f IS ONTO $\exists a \in A$ WITH $B = f(a)$

CASE 1 $a \in B$, THEN BY DEF OF B, $a \notin f(a) = B$, SO $a \notin B \Rightarrow \Leftarrow$

CASE 2 $a \notin B$ THEN $a \notin f(a)$ BUT SINCE ALSO $a \in A$, a SATISFIES ALL THE PROPERTIES OF B AND SO $a \in B$ $\Rightarrow \Leftarrow$

IN BOTH CASES WE HAVE A CONTRADICTION, AND THEREFORE f CANNOT EXIST, THAT IS THERE IS NO ONTO FUNCTION $f : A \to P(A)$.