SOLUTIONS

MATH 453 — FINAL EXAM

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Name: ______________________________

Instructions: Welcome to your final exam! You have 24 hours to take this exam, for a total of 100 points. Write in full sentences whenever you can and try to be as precise as you can. If you need to continue your work on the back of a page, please indicate that you're doing so, or else your work may be discarded. May your luck be noncharacteristic! :)

Honor Code: I promise not to communicate with anyone about this exam unless everyone in my group has already taken it. I also promise not to use any books or notes or cheat sheets or personal electronic devices (including calculators).

Signature: ______________________________

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Date: Saturday, May 13 – Sunday, May 21.

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1. *(10 points)* Prove that solutions to Poisson's equation are unique:

$$\begin{cases} -\Delta u = f & \text{in } W \\ u = g & \text{on } \partial W \end{cases}$$

(a) Using energy methods
(b) Using the (weak or strong) maximum principle

SUPPOSE THAT U & V ARE TWO SOLUTIONS AND LET $\tilde{w} = U - V$

THEN $\tilde{w}$ SOLVES $\begin{cases} -\Delta \tilde{w} = 0 & \text{IN } W \\ \tilde{w} = 0 & \text{ON } \partial W \end{cases}$ $(*)$

(a) ENERGY METHODS

MULTIPLY $(*)$ BY $\tilde{w}$ AND INTEGRATE BY PARTS:

$$0 = \int_W (-\Delta \tilde{w})\, \tilde{w} = \int_{\partial W} -\frac{\partial \tilde{w}}{\partial \nu} \underbrace{\tilde{w}}_{= 0} + \int_W |\nabla \tilde{w}|^2 = \int_W |\nabla \tilde{w}|^2$$

HENCE $\nabla \tilde{w} \equiv 0$, AND SO $\tilde{w} \equiv C$

BUT SINCE $\tilde{w} \equiv 0$ ON ∂W, WE GET $C = 0$

AND SO $\tilde{w} \equiv 0$, SO $U \equiv V$ ✓

(b) MAXIMUM PRINCIPLE

BY THE WEAK MAXIMUM PRINCIPLE, WE HAVE

$$\max_{\overline{W}} \tilde{w} = \max_{\partial W} \underbrace{\tilde{w}}_{= 0} = 0 \quad \text{SO} \quad \tilde{w} \leq 0 \quad \text{ON } \overline{W}$$

AND $$\min_{\overline{W}} \tilde{w} = \min_{\partial W} \underbrace{\tilde{w}}_{= 0} = 0 \quad \text{SO} \quad \tilde{w} \geq 0 \quad \text{ON } \overline{W}$$

AND HENCE $\tilde{w} \equiv 0$ ON $\overline{W}$

SO $U \equiv V$ ✓

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2. *(10 points)* State and prove D'Alembert's formula for the solution of the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R} \times \{t=0\} \\ u_t(x,0) = h(x) & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

Note: You may use **without proof** that the solutions of the nonhomogeneous transport equation

$$\begin{cases} u_t + b \cdot Du = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

are of the form

$$u(x,t) = g(x - tb) + \int_0^t f(x + (s-t)b, s)\,ds$$

D'ALEMBERT'S FORMULA $\quad U(x,t) = \frac{1}{2}\left(g(x-t) + g(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$

PROOF $\quad$ NOTICE THAT $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) U = U_{tt} - U_{xx} = 0$

1) SO LET $V = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) U = U_t - U_x$

THEN $\quad V_t + V_x = (U_t - U_x)_t + (U_t - U_x)_x = U_{tt} - \cancel{U_{xt}} + \cancel{U_{tx}} - U_{xx}$
$= U_{tt} - U_{xx} = 0$

$V_t + V_x = 0$

2) SO V SATISFIES A TRANSPORT EQUATION WITH $b = 1$, $f \equiv 0$ AND SO

$V(x,t) = a(x-t) + \int_0^t 0 = a(x-t)$ FOR SOME FUNCTION a $\mathbb{R} \to \mathbb{R}$

3) BUT THEN $\quad U_t - U_x = V \Rightarrow U_t - U_x = a(x-t)$, SO

U SATISFIES A TRANSPORT EQUATION WITH $b = -1$, $f(x,t) = a(x-t)$

HENCE $\quad U(x,t) = \tilde{a}(x+t) + \int_0^t a(x + (s-t)(-1) - s)\,ds$

$U(x,t) = \tilde{a}(x+t) + \int_0^t a(\underbrace{x+t-2s}_{y})\,ds = \tilde{a}(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy$

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4) Now $U(x,0) = \tilde{a}(x+0) + \int_x^x \cdots = \tilde{a}(x) = g(x)$

so $\tilde{a} = g$

And $U_t(x,t) = g'(x+t) + \frac{1}{2}\left(a(x+t) + a(x-t)\right)$

$U_t(x,0) = g'(x) + \frac{1}{2}\left(2a(x)\right) = g'(x) + a(x)$

Hence $a(x) = h(x) - g'(x)$

5) $U(x,t) = g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} h(y) - g'(y)\,dy$

$= g(x+t) - \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$

$$\boxed{U(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy}$$

3. *(10 points)* Suppose u is a minimizer of

$$I[w] = \int_W L(Dw(x), w(x), x)dx$$

among all smooth functions $w : \overline{W} \to \mathbb{R}$ satisfying the boundary condition $w = g$ on ∂W, where $L = L(p, z, x)$ and $g = g(x)$ are given.

What PDE must u satisfy? Explain how you obtained your answer.

CLAIM U MUST SATISFY

$$\begin{cases} -\sum_{i=1}^{N} L_{p_i}(DU(x), U(x), x) + L_z(DU(x), U(x), x) = 0 & \text{IN } W \\ U = g & \text{ON } \partial W \end{cases}$$

PROOF LET U BE A MINIMIZER AND LET $V \in C_c^\infty(W)$

CONSIDER $i(\tau) = I[U + \tau V] = \int_W L(DU(x) + \tau DV(x), U(x) + \tau V(x), x)\, dx$

SINCE U IS A MINIMIZER, $i'(0) = 0$.

BUT $i'(\tau) = \int_W \sum_{i=1}^{N} L_{p_i}(DU + \tau DV, U + \tau V, x) V_{x_i} + L_z(DU + \tau DV, U + \tau V, x) V \, dX$

$V \equiv 0$ ON ∂W $\xrightarrow{\text{IBP}}$ $= \int_W \sum_{i=1}^{N} -(L_{p_i}(DU + \tau DV, U + \tau V, x))_{x_i} V + L_z(DU + \tau DV, U + \tau V, x) V \, dx$

SINCE $i'(0) = 0$, WE GET

$$\int_W \left[\left(\sum_{i=1}^{N} (-L_{p_i}(DU, U, x))_{x_i} \right) + L_z(DU, U, x) \right] V \, dx = 0 \quad \forall V \in C_c^\infty(W)$$

BUT SINCE V IS ARBITRARY, WE GET

$$\boxed{\sum_{i=1}^{N} (-L_{p_i}(DU, U, x))_{x_i} + L_z(DU, U, x) = 0}$$

(AND $U = g$ ON ∂W SINCE $W = g$ $\forall W$)

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4. *(10 points)* Find a solution of the following heat equation on the half-line:

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0,\infty)\times(0,\infty) \\ u(x,0) = g(x) & \text{on } (0,\infty)\times\{t=0\} \\ u_x(0,t) = 0 & \text{on } \{x=0\}\times(0,\infty) \end{cases}$$

Note: I'd like to remind you that the fundamental solution for the heat equation in one dimension is $\Phi(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$

1) CONSIDER THE EVENIFICATION $\tilde{g}$ OF g DEFINED BY :

$$\tilde{g}(x) = \begin{cases} g(x) & x \geq 0 \\ g(-x) & x < 0 \end{cases}$$

AND CONSIDER THE FOLLOWING HEAT EQUATION ON THE WHOLE REAL LINE :

$$\begin{cases} \tilde{U}_t - \tilde{U}_{xx} = 0 & \text{IN } \mathbb{R}\times(0,\infty) \\ \tilde{U}(x,0) = \tilde{g}(x) & \text{ON } \mathbb{R}\times\{t=0\} \end{cases}$$

FROM LECTURE, WE KNOW THAT A SOLUTION IS GIVEN BY

$$\tilde{U}(x,t) = \Phi * \tilde{g} = \int_{-\infty}^{\infty} \Phi(x-y)\,\tilde{g}(y)\,dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\,\tilde{g}(y)\,dy$$

2) NOW IF $x > 0$, LET $U(x,t) = \tilde{U}(x,t)$

THEN $U_t = U_{xx}$ B/C $\tilde{U}$ SATISFIES $\tilde{U}_t = \tilde{U}_{xx}$

AND IF $x > 0$ $U(x,0) = \tilde{U}(x,0) = \tilde{g}(x) = g(x)$ (SINCE $\tilde{g} = g$ IF $x > 0$)

AND $U_x(x,t) = \tilde{U}_x(x,t) = \int_{-\infty}^{\infty} \Phi_x(x-y)\,\tilde{g}(y)\,dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} -\frac{2(x-y)}{4t} e^{-\frac{(x-y)^2}{4t}}\,\tilde{g}(y)\,dy$

SO $U_x(0,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \underbrace{\frac{2y}{4t} e^{\frac{-y^2}{4t}}}_{\text{ODD}} \underbrace{\tilde{g}(y)}_{\text{EVEN}}\,dy = 0$

SO U INDEED SATISFIES OUR EQUATION

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3) FINALLY, LET'S WRITE u IN TERMS OF g ONLY:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi t}} \left(\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} g(-y)\,dy + \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4t}} g(y)\,dy \right)$$

$y \mapsto -y$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} \left(e^{-\frac{(x+y)^2}{4t}} + e^{-\frac{(x-y)^2}{4t}} \right) g(y)\,dy}$$

5. *(10 points)* Solve using characteristics:

$$\begin{cases} (u_{x_1})^3 + (u_{x_2})^3 = 2 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x_1, 0) = x_1 & \text{on } \mathbb{R} \times \{x_2 = 0\} \end{cases}$$

Note: The characteristic ODE are given by:

$$\begin{cases} \mathbf{x}'(s) = D_p F \\ z'(s) = \mathbf{p}(s) \cdot D_p F \\ \mathbf{p}'(s) = -D_x F - (F_z)\mathbf{p}(s) \end{cases}$$

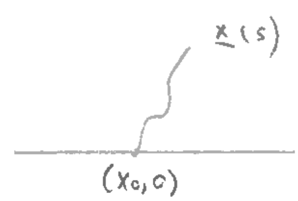
STEP 1 CHARACTERISTIC ODE

HERE $F(p, z, x) = F(p_1, p_2, z, x_1, x_2) = (p_1)^3 + (p_2)^3 - 2$

$D_p F = (3p_1^2, 3p_2^2)$, $D_x F = 0$, $F_z = 0$

HENCE $\begin{cases} \underline{x}'(s) = D_p F \\ z'(s) = p \cdot D_p F \\ p'(s) = -D_x F - (F_z) p \end{cases}$ BECOMES $\begin{cases} x_1'(s) = 3(p_1(s))^2 \\ x_2'(s) = 3(p_2(s))^2 \\ z'(s) = 3p_1^3 + 3p_2^3 = 3 \times 2 = 6 \\ p_1'(s) = 0 \\ p_2'(s) = 0 \end{cases}$ PDE! ✓

$$\Longrightarrow \begin{cases} x_1'(s) = 3A^2 \Rightarrow x_1(s) = 3A^2 s + D \\ x_2'(s) = 3B^2 \Rightarrow x_2(s) = 3B^2 s + E \\ z(s) = 6s + C \\ p_1(s) = A \\ p_2(s) = B \end{cases}$$



STEP 2 FIND A, B, C, D, E

$x_0 = x_1(0) = 3A^2(0) + D \Rightarrow \underline{D = x_0}$

$0 = x_2(0) = 3B^2(0) + E \Rightarrow \underline{E = 0}$

$z(0) = 6(0) + C = C$ BUT ALSO $z(0) = u(x_0, 0) = x_0 \Rightarrow \underline{C = x_0}$

$p_1(0) = A$ BUT ALSO $p_1(0) = u_{x_1}(x_0, 0) = (x_0)_{x_1} = 1 \Rightarrow \underline{A = 1}$

$p_1^3(0) + p_2^3(0) = 2 \Rightarrow 1 + B^3 = 2 \Rightarrow B^3 = 1 \Rightarrow \underline{B = 1}$

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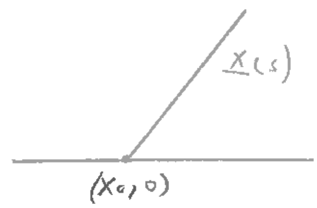
HENCE
$$\begin{cases} x_1(s) = 3s + x_0 \\ x_2(s) = 3s \\ z(s) = 6s + x_0 \\ p_1(s) = 1 \\ p_2(s) = 1 \end{cases}$$

CHARACTERISTICS ARE STRAIGHT LINES!



STEP 3 FIND x_0 & s^*

GIVEN $x = (x_1, x_2)$, FIND s^* AND x_0 SUCH THAT $x = \underline{x}(s^*)$

THIS GIVES
$$\begin{cases} x_1 = 3s^* + x_0 \\ x_2 = 3s^* \end{cases} \Rightarrow \begin{cases} x_0 = x_1 - x_2 \\ s^* = \frac{x_2}{3} \end{cases}$$

STEP 4 HENCE $u(x_1, x_2) = z(s^*) = 6s^* + x_0 = 2x_2 + x_1 - x_2 = x_1 + x_2$

$$\boxed{u(x_1, x_2) = x_1 + x_2}$$

6. Solve using separation of variables:

$$\begin{cases} u_t = u_{xx} & 0 < x < 1, t > 0 \\ u_x(0,t) = 0,\ U_x(1,t)=0 & t > 0 \\ u(x,0) = \cos(2\pi x) & 0 < x < 1 \end{cases}$$

[redacted]

STEP 1 SEPARATION OF VARIABLES

SUPPOSE $U(x,t) = X(x)T(t)$, PLUG INTO $U_t = U_{xx}$

$$(XT)_t = (XT)_{xx}$$

$$XT' = X''T$$

$$\underbrace{\frac{X''}{X}}_{\text{ONLY DEPENDS ON } x} = \underbrace{\frac{T'}{T}}_{\text{ONLY DEPENDS ON } t} = \lambda \implies \begin{cases} X'' = \lambda X \\ T' = \lambda T \end{cases}$$

STEP 2 EQUATION FOR X

Now $U(x,t) = XT$, $U_x(x,t) = X'(x)T(t)$, so $U_x(0,t) = X'(0)\cancel{T(t)} = 0$

$\implies X'(0) = 0$

$U_x(1,t) = X'(1)T(t) = 0$

$\implies X'(1) = 0$

WHENCE $\begin{cases} X'' = \lambda X \\ X'(0) = 0,\ X'(1) = 0 \end{cases}$

CASE 1 $\lambda > 0$, THEN $\lambda = \omega^2$, $\omega > 0$, so $X'' = \lambda X \implies X'' = \omega^2 X$

$$\implies X(x) = Ae^{\omega x} + Be^{-\omega x}$$

BUT THEN $X'(x) = A\omega e^{\omega x} - B\omega e^{-\omega x}$

$X'(0) = A\omega - B\omega = 0 \implies A = B$

$X'(1) = \cancel{A\omega} e^{\omega} - \cancel{A\omega} e^{-\omega} = 0 \implies e^{\omega} = e^{-\omega} \implies \omega = 0 \implies\Leftarrow$

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CASE 2 $\lambda = 0$, THEN $X'' = 0 \Rightarrow X(x) = Ax + B$

$X'(0) = A = 0$, SO $X(x) = B$, BUT THEN AUTOMATICALLY $X'(1) = 0$, SO $X(x) = B$.

THIS IS A LEGIT SOLUTION!

CASE 3 $\lambda < 0$, THEN $\lambda = -\omega^2$, $\omega > 0$, SO $X'' = \lambda X \Rightarrow X'' = -\omega^2 X$

$\Rightarrow X(x) = A\cos(\omega x) + B\sin(\omega x)$

$X'(x) = -A\omega\sin(\omega x) + B\omega\cos(\omega x)$

$X'(0) = -A\omega\, 0 + B\omega = 0 \Rightarrow B = 0 \Rightarrow X(x) = A\cos(\omega x)$

$X'(1) = -A\omega\sin(\omega) = 0 \Rightarrow \sin(\omega) = 0 \Rightarrow \omega = \pi m \quad (m = 1, 2, 3, \dots)$

WHICH GIVES $\lambda = -(\pi m)^2$ AND SOLUTIONS $X_m(x) = A_m\cos(\pi m x)$

STEP 3 EQUATION FOR T $\quad T' = \lambda T \Rightarrow T(t) = Ae^{\lambda t}$

FOR $\lambda = 0$ (CASE 2), THIS GIVES $T(t) = A$

FOR $\lambda = -(\pi m)^2$ (CASE 3), THIS GIVES $T(t) = A_m e^{-(\pi m)^2 t}$

STEP 4
$$U(x,t) = \underbrace{A_0}_{X}\, \underbrace{e^{-(\pi m)^2 t}}_{T} + \sum_{m=1}^{\infty} A_m \underbrace{e^{-(\pi m)^2 t}}_{T} \underbrace{\cos(\pi m x)}_{X}$$

$$U(x,0) = A_0 + \sum_{m=1}^{\infty} A_m\cos(\pi m x) = \cos(2\pi x) \Rightarrow A_i = 0 \text{ FOR ALL } i,$$

EXCEPT $A_2 = 1$

STEP 5 CONCLUSION $\quad U(x,t) = A_2 e^{-(\pi 2)^2 t}\cos(2\pi x)$

$$\boxed{U(x,t) = e^{-4\pi^2 t}\cos(2\pi x)}$$

7. *(10 points)* Let $u(x) = \frac{1}{|x|}$ on $B = B(0,1)$ (the open unit ball in $\mathbb{R}^n$). WITH $N \geq 5$

(a) Find the weak derivative Du of u and prove it is the weak derivative of u. Beware of the singularity at 0.

(b) Is u in $H^1(B)$? Why or why not?

(a) GUESS $DU(x) = \left(-\frac{1}{|x|^2}\right)\frac{x}{|x|} = \frac{-x}{|x|^3}$

PROOF LET $\varphi \in C_c^\infty(B)$ BE ARBITRARY, THEN $\forall i = 1, \ldots, N$ AND $\forall \varepsilon > 0$

$$\int_B U \varphi_{x_i}\, dx = \underbrace{\int_{B(0,\varepsilon)} U \varphi_{x_i}\, dx}_{A} + \underbrace{\int_{B \setminus B(0,\varepsilon)} U \varphi_{x_i}\, dx}_{B}$$

NOW $$|A| \leq \int_{B(0,\varepsilon)} |U|\, \underbrace{|\varphi_{x_i}|}_{\leq C} = \int_{B(0,\varepsilon)} \frac{1}{|x|} = \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{r}\, dS(y)\, dr$$

$$= \int_0^\varepsilon \frac{1}{r}\, \alpha(N) N r^{N-1}\, dr = \alpha(N) N \int_0^\varepsilon r^{N-2}\, dr =$$

$$= \alpha(N) N \left[\frac{r^{N-1}}{N-1}\right]_0^\varepsilon = N\alpha(N) \frac{\varepsilon^{N-1}}{N-1} \longrightarrow 0 \quad \text{SINCE } N > 1$$

NOW SINCE U DOES NOT HAVE A SINGULARITY ON $B \setminus B(0,\varepsilon)$, WE CAN LEGITIMATELY INTEGRATE BY PARTS ON $B \setminus B(0,\varepsilon)$ AND WE GET

$$B = \underbrace{\int_{\partial B(0,\varepsilon)} U \varphi_{x_i} \nu^i}_{C} - \underbrace{\int_{B \setminus B(0,\varepsilon)} \left(\frac{-x_i}{|x|^3}\right) \varphi}_{D}$$

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Now $|\textcircled{C}| \leq \int_{\partial B(0,\epsilon)} |u| \, \underbrace{|\varphi_{x_i}| \, |\nu^i|}_{\leq C}$

$$= \int_{\partial B(0,\epsilon)} \frac{1}{\epsilon}$$

$$= \frac{N\alpha(N)\epsilon^{N-1}}{\epsilon} = N\alpha(N)\epsilon^{N-2} \longrightarrow 0 \quad \text{SINCE } N > 2$$

ALSO FOR $\textcircled{D}$, NOTICE THAT $\left| \frac{-x_i}{|x|^3} \varphi \right| \leq C \frac{|x|}{|x|^3} = \frac{1}{|x|^2}$

AND $\int_B \frac{1}{|x|^2} = \int_0^1 \int_{\partial B(0,r)} \frac{1}{r^2} = \int_0^1 \frac{1}{r^2} N\alpha(N) r^{N-1} dr$

$$= \int_0^1 N\alpha(N) r^{N-3} = \left[\frac{r^{N-2}}{N-2} \right]_0^1 = \frac{1}{N-2} < \infty$$

HENCE BY THE DCT WE CAN PASS THROUGH THE LIMIT AS $\epsilon \to 0$ IN $\textcircled{D}$, AND WE INDEED GET

$$\int_B u \varphi_{x_i} dx = -\int_B \left(\frac{-x_i}{|x|^3} \right) \varphi \, dx \quad \text{, SO } Du = -\frac{x}{|x|^3}$$

(b) ALL WE NEED TO VERIFY IS THAT $u \in L^2(B)$ AND $Du \in L^2(B)$

BUT $\int_B |u|^2 = \int_B \frac{1}{|x|^2} = \frac{1}{N-2}$ (SEE ABOVE) $< \infty$

AND $\int_B |Du|^2 = \int_B \frac{|x|^2}{|x|^6} = \int_B \frac{1}{|x|^4} = \int_0^1 \int_{\partial B(0,r)} \frac{1}{r^4} = \int_0^1 N\alpha(N) r^{N-1-4} = N\alpha(N) \left[\frac{r^{N-4}}{N-4} \right]_0^1$

$$= \frac{N\alpha(N)}{N-4} < \infty \quad \text{SINCE } N \geq 5$$

THEREFORE u IS INDEED IN $H^1(B)$

8. *(10 points)* Suppose that $u = u(x,t)$ is a solution of

$$\begin{cases} u_t + \Delta u = 0 & \text{in } W \times [0,T] \\ u(x,t) = 0 & \text{on } \partial W \\ u(x,T) = h(x) & \text{on } W \times \{t = T\} \end{cases}$$

and that $v = v(x,t)$ is a solution of

$$\begin{cases} v_t - \Delta v = 0 & \text{in } W \times [0,T] \\ v(x,t) = 0 & \text{on } \partial W \\ v(x,0) = 0 & \text{on } W \times \{t = 0\} \end{cases}$$

Use energy methods to show that

$$\int_W v(x,T)h(x)dx = 0$$

LET $E(t)$ BE THE ENERGY

$$E(t) = \int_W U(x,t)\, V(x,t)\, dx$$

THEN

$$E'(t) = \int_W (U_t)V + UV_t \, dx$$

$$= \int_W (-\Delta U)V + U(\Delta V)\, dx$$

(IBP)

$$= \int_{\partial W} -\left(\frac{\partial U}{\partial \nu}\right)\underbrace{V}_{=0} + \cancel{\int_W DU \cdot DV} + \int_{\partial W} \underbrace{U}_{=0}\left(\frac{\partial V}{\partial \nu}\right) - \cancel{\int_W DU \cdot DV}$$

$$= 0$$

HENCE E IS CONSTANT AND

$$\int_W h(x)\, V(x,T)\, dx = \int_W U(x,T)\, V(x,T)\, dx = E(T) = E(0) = \int_W U(x,0)\underbrace{V(x,0)}_{=0} = 0$$

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9. *(10 points)*

(a) Prove the following variant of Cauchy's inequality (called Cauchy's inequality with ϵ):

For any $a, b \in \mathbb{R}$, and any $\epsilon > 0$, we have

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

Hint: Write $ab = (\sqrt{2\epsilon}a)\left(\frac{b}{\sqrt{2\epsilon}}\right)$ and use Cauchy's inequality.

(b) Use (a) and Poincaré's inequality to show that if [redacted] $u \in C_c^\infty(W)$ is a [redacted] solution of Poisson's equation

$$\begin{cases} -\Delta u = f & \text{in } W \\ u = 0 & \text{on } \partial W \end{cases}$$

then there is a constant $C > 0$ depending *only* on W and the dimension n such that

$$\int_W (u(x))^2 dx \leq C\left(\int_W (f(x))^2 dx\right)$$

This shows that the 'energy' of u is uniformly bounded by the energy of f.

(a)

$$ab = (\sqrt{2\varepsilon}a)\left(\frac{b}{\sqrt{2\varepsilon}}\right)$$

$$\leq \frac{1}{2}(\sqrt{2\varepsilon}a)^2 + \frac{1}{2}\left(\frac{b}{\sqrt{2\varepsilon}}\right)^2 \quad \checkmark \text{ CAUCHY}$$

$$= \frac{1}{2} 2\varepsilon a^2 + \frac{1}{2}\frac{b^2}{2\varepsilon}$$

$$= \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

HENCE $ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$

(b) MULTIPLY $-\Delta u = f$ BY u AND IBP ON W

$$\int_W (-\Delta u) u = \int_W f u$$

BUT $\int_W (-\Delta u) u = \overbrace{\int_{\partial W} -\left(\frac{\partial u}{\partial \nu}\right) u}^{0} + \int_W |Du|^2 = \int_W |Du|^2$

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HENCE
$$\int_W |DU|^2 = \int_W fU$$

ON THE OTHER HAND, BY POINCARÉ, WE KNOW THAT SINCE $U \in C_c^\infty(W)$

$$\int_W |U|^2 \leq C \int_W |DU|^2$$ FOR SOME C, WHERE C ONLY DEPENDS ON W AND N

HENCE
$$\int_W |DU|^2 \geq \frac{1}{C} \int_W |U|^2$$ AND SO

$$\frac{1}{C} \int_W |U|^2 \leq \int_W fU$$

HOWEVER, NOW BY (a) WE GET $\forall \varepsilon > 0$, $$\int_W fU = \int_W \overset{a}{U} \overset{b}{f} \leq \varepsilon \int_W U^2 + \frac{1}{4\varepsilon} \int_W f^2$$

PLUGGING THIS INTO THE ABOVE, WE GET

$$\frac{1}{C} \int_W |U|^2 \leq \varepsilon \int_W U^2 + \frac{1}{4\varepsilon} \int_W f^2$$

AND SO $$\left(\frac{1}{C} - \varepsilon\right) \int_W U^2 \leq \frac{1}{4\varepsilon} \int_W f^2$$

<u>NOW CHOOSE</u> ε SUCH THAT $C' := \frac{1}{C} - \varepsilon > 0$ (SAY $\varepsilon = \frac{1}{2C}$),

THEN WE GET $$\int_W U^2 \leq \frac{1}{4\varepsilon C'} \int_W f^2$$, SO IF YOU LET $C := \frac{1}{4\varepsilon C'}$, WE GET

$$\boxed{\int_W U^2 \leq C \int_W f^2}$$

10. *(10 points)* Suppose that u solves $\Delta u = 0$ on $\overline{B(0,R)}$ (for some fixed $R > 0$) and $u(0) = 0$. Our goal in this problem is to figure out how fast u goes to 0 at $x = 0$.

For $0 < r \leqslant R$, define the functions

$$a(r) := \frac{1}{r^{n-1}} \int_{\partial B(0,r)} |u(y)|^2 \, dS(y)$$

$$b(r) := \frac{1}{r^{n-2}} \int_{B(0,r)} |Du(x)|^2 \, dx$$

We define the *radial* derivative of u to be $u_r(x) := Du \cdot \frac{x}{|x|}$.

(a) Show that the following identities hold:

$$b(r) = \frac{1}{r^{n-2}} \int_{\partial B(0,r)} u(y) u_r(y) dS(y)$$

$$a'(r) = \frac{2}{r^{n-1}} \int_{\partial B(0,r)} u(y) u_r(y) dS(y)$$

Hint: For the first, use the definition of b and integrate by parts.

One can also show that (do **NOT** do this; it uses what's called Noether's theorem):

$$b'(r) = \frac{2}{r^{n-2}} \int_{\partial B(0,r)} (u_r(y))^2 dS(y)$$

(b) Deduce from (a) that $(b(r))^2 \leq \frac{r}{2} a(r) b'(r)$

(c) Define $f(r) := \frac{b(r)}{a(r)}$ (the frequency function) and deduce from (a) and (b) that $f'(r) \geq 0$. This is called Almgren's monotinicity formula (who was Frank Morgan's PhD advisor!)

(d) Deduce from (b) and (c) that $\frac{a'(r)}{a(r)} \leq \frac{\beta}{r}$ and therefore that $a(r) \geq \gamma r^{\beta}$, where $\beta := 2\frac{b(R)}{a(R)}$, $\gamma := \frac{a(R)}{R^{\beta}}$. Conclude by finding a constant C (in terms of [redacted]) and a real number $s > 0$ (in terms of n and β) such that (yes, the whole ball)

σ, β, n

$$\int_{B(0,r)} (u(x))^2 \, dx \geq C r^s$$

So, in terms of energies, u goes to 0 at rate at least r^s.

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(a) 1) $b(r) = \frac{1}{r^{N-2}} \int_{B(0,r)} Du(x) \cdot Du(x)\, dx$

$$\overset{\text{IBP}}{=} \frac{1}{r^{N-2}} \int_{\partial B(0,r)} u(x) \frac{\partial u}{\partial \nu}(x)\, ds(x) - \frac{1}{r^{N-2}} \int_{B(0,r)} \underbrace{\Delta u(x)}_{\overset{\shortparallel}{0}} u(x)\, dx$$

$\nu = \frac{x}{|x|}$

$$= \frac{1}{r^{N-2}} \int_{\partial B(0,r)} u(x) \underbrace{Du(x) \cdot \frac{x}{|x|}}_{u_r(x)}\, ds(x)$$

$$= \frac{1}{r^{N-2}} \int_{\partial B(0,r)} u(x)\, u_r(x)\, ds(x)$$

2) CHANGING VARIABLES $y' = \frac{y}{r}$, WE GET

$$a(r) = \frac{1}{\cancel{r^{N-1}}} \int_{\partial B(0,1)} |u(ry')|^2 \cancel{r^{N-1}}\, ds(y') = \int_{\partial B(0,1)} |u(ry)|^2\, ds(y)$$

HENCE $a'(r) = \int_{\partial B(0,1)} 2u(ry)\, Du(ry) \cdot y\, ds(y)$ $\quad y' = ry$

$$= \frac{1}{r^{N-1}} \int_{\partial B(0,r)} 2u(y) \underbrace{Du(y) \cdot \frac{y}{r}}_{u_r(y)}\, ds(y) \quad (\text{SINCE } r = |y|)$$

$$= \frac{2}{r^{N-1}} \int_{\partial B(0,r)} u(y)\, u_r(y)\, ds(y)$$

(b) USING CAUCHY-SCHWARZ, WE GET

$$b(r) \le \frac{1}{r^{N-2}} \left(\int_{\partial B(0,r)} |u(x)|^2 \right)^{\frac{1}{2}} \left(\int_{\partial B(0,r)} |u_r(x)|^2 \right)^{\frac{1}{2}}$$

$$= \frac{1}{r^{N-2}} \left(r^{N-1}\right)^{\frac{1}{2}} \left(\frac{r^{N-2}}{2}\right)^{\frac{1}{2}} \left(\frac{1}{r^{N-1}} \int_{\partial B(0,r)} |u|^2 \right)^{\frac{1}{2}} \left(\frac{2}{r^{N-2}} \int_{\partial B(0,r)} |u_r(x)|^2 \right)^{\frac{1}{2}}$$

$$= \left(\frac{r}{2}\right)^{\frac{1}{2}} \left(\frac{1}{r^{N-1}} \int_{\partial B(0,r)} |u|^2 \right)^{\frac{1}{2}} \left(\frac{2}{r^{N-2}} \int_{\partial B(0,r)} |u_r|^2 \right)^{\frac{1}{2}}$$

BY (a)

$$= \left(\frac{r}{2}\right)^{\frac{1}{2}} \left(a(r)\right)^{\frac{1}{2}} \left(b'(r)\right)^{\frac{1}{2}}$$

So $b(r) \le \left(\frac{r}{2}\right)^{\frac{1}{2}} (a(r))^{\frac{1}{2}} (b'(r))^{\frac{1}{2}}$ AND SQUARING BOTH SIDES, WE GET

$$(b(r))^2 \le \left(\frac{r}{2}\right) (a(r)) (b'(r))$$

(c)

$$f'(r) = \frac{b'(r)\,a(r) - b(r)\,a'(r)}{(a(r))^2}$$

BY (b)

$$\ge \frac{\frac{2}{r}\frac{(b(r))^2}{\cancel{a(r)}}\,\cancel{a(r)} - b(r)\,a'(r)}{(a(r))^2}$$

$$= \frac{\frac{2}{r}(b(r))^2 - b(r)\,a'(r)}{a(r)} = 0$$

BUT NOTICE THAT $a'(r) = \frac{2}{r}\left(\frac{1}{r^{N-1}} \int_{\partial B(0,r)} u\,u_r \right) = \frac{2}{r} b(r)$

so $a'(r)\,b(r) = \frac{2}{r}(b(r))^2 \Rightarrow \frac{2}{r}(b(r))^2 - a'(r)\,b(r) = 0$

(d) IN PARTICULAR, f IS NONDECREASING AND

$$f(r) \le f(R)$$

$$\Rightarrow \quad \frac{f(r)}{a(r)} \le \frac{f(R)}{a(R)} \qquad a'(r) = \frac{2}{r} f(r)$$

$$\frac{\frac{r}{2} a'(r)}{a(r)} \le \frac{f(R)}{a(R)}$$

$$\frac{a'(r)}{a(r)} \le \underbrace{\frac{2 f(R)}{a(R)}}_{=\beta} \frac{1}{r}$$

$$\Rightarrow \frac{a'(r)}{a(r)} \le \frac{\beta}{r}, \quad \text{so} \quad (\ln(a(r)))' \le \frac{\beta}{r} \qquad (a > 0)$$

HENCE $$\int_r^R (\ln(a(s)))'\, ds \le \int_r^R \frac{\beta}{s}\, ds$$

$$\ln(a(R)) - \ln(a(r)) \le \beta\,(\ln(R) - \ln(r))$$

$$\ln\left(\frac{a(R)}{a(r)}\right) \le \beta \ln\left(\frac{R}{r}\right)$$

$$\Rightarrow \frac{a(R)}{a(r)} \le e^{\beta \ln\left(\frac{R}{r}\right)} = \left(\frac{R}{r}\right)^{\beta}$$

$$\Rightarrow a(r) \ge \frac{r^{\beta}}{R^{\beta}} a(R) = \overbrace{\frac{a(R)}{R^{\beta}}}^{\gamma} r^{\beta} = \gamma r^{\beta}$$

WHICH MEANS, BY DEFINITION OF a, THAT

$$\frac{1}{r^{N-1}} \int_{\partial B(0,r)} |u(y)|^2 \, dS(y) \geq \gamma r^{\beta}$$

$$\Rightarrow \int_{\partial B(0,r)} |u(y)|^2 \, dS(y) \geq \gamma r^{\beta+N-1}$$

FINALLY,

POLAR COORDS

$$\int_{B(0,r)} |u(x)|^2 \, dx = \int_0^r \int_{\partial B(0,s)} |u(s)|^2 \, dS(s)$$

$$\geq \int_0^r \gamma s^{\beta+N-1} \, ds$$

$$= \gamma \left[\frac{s^{\beta+N}}{\beta+N} \right]_0^r$$

$$= \frac{\gamma r^{\beta+N}}{\beta+N}$$

WHENCE, IF YOU LET $C := \frac{\gamma}{\beta+N}$ AND $s := \beta+N$, THEN YOU GET

$$\boxed{\int_{B(0,r)} |u(x)|^2 \, dx \geq C r^{s}}$$