SOLUTIONS

MATH 200 – FINAL EXAM

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Name: ______________________________

Instructions: Welcome to the Final Exam! You have 150 minutes (= 2h30) to take this exam, for a total of 100 points. **Do not open the exam until you're instructed to do so**. Remember that you are not only graded on the correctness of your answer, but also on the clarity and completeness of your proofs. Write in complete sentences whenever you can. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. Good luck, and may the odds be in your favor! :)

Honor Code: I promise not to communicate or collaborate with anyone during the exam, and I will not use any books or notes or cheat sheets or personal electronic devices (**including calculators**).

Signature: ______________________________

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total		100

Date: Thursday, December 15, 2016.

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1. *(20 points, 2 pts each)*

Label the following statements as **TRUE (T)** or **FALSE (F)**. Any correct answer gives you 2 points, and any incorrect answer gives you 0 points. You do **NOT** get negative points for an incorrect answer, and you do **NOT** need to justify your answers.

F (a) The probability that the answer to exactly 2 of the questions in this problem is T is $\frac{2}{10}$.

T (b) $A \times B$ always has the same cardinality as $B \times A$, even if A and B don't necessarily have the same cardinality.

F (c) If a graph G has 4 vertices, each with degree $1, 2, 3, 4$, then G has an Euler circuit.

T (d) If events A and B are independent, then A^c and B^c are independent as well.

F (e) There is a full binary tree with 4 leaves and 6 branches.

F (f) If a, b, c are positive integers and a divides bc, then a divides b or a divides c.

T (g) The following identity holds for all $n \in \mathbb{N}$:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

T (h) There exists a graph with 4 vertices of degrees $0, 1, 2, 3$ respectively

F (i) If $\exists y \forall x P(x, y)$, then $\exists x \forall y P(x, y)$

F (j) ~~[illegible]~~ ANY function from $\{1, 2, 3, 4, 5\}$ to $\{1, 2\}$ must be onto.

EXPLANATIONS ON THE NEXT PAGE

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EXPLANATIONS (OPTIONAL)

(F) (a) THE PROBABILITY IS $\binom{10}{2}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7 = \binom{10}{2}\left(\frac{1}{2}\right)^{10} = \frac{90}{2^{11}} \neq \frac{1}{5}$

(T) (b) THE FUNCTION $f: A \times B \to B \times A$ $f(a,b) = (b,a)$ IS A BIJECTION

(F) (c) FOR A GRAPH TO HAVE AN EULER CIRCUIT, THE DEGREE OF EACH VERTEX MUST BE EVEN.

(T) (d) $P(A^c \cap B^c) = 1 - P((A^c \cap B^c)^c) = 1 - P(A \cup B)$ (COMPLEMENT; DE MORGAN; INCLUSION-EXCLUSION)

$= 1 - P(A) - P(B) + P(A \cap B)$ (A & B ARE INDEP)

$= 1 - P(A) - P(B) + P(A)P(B)$

$= 1 - P(A) + P(B)(P(A) - 1)$

$= (1 - P(A))(1 - P(B))$ (COMPLEMENT)

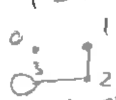
$= P(A^c)P(B^c)$

(F) (e) A FULL BINARY TREE WITH K BRANCHES MUST HAVE K+1 LEAVES

(F) (f) $4 \mid 36 = 6 \times 6$, BUT $4 \nmid 6$ AND $4 \nmid 6$; a AND b MUST BE RELATIVELY PRIME!

(T) (g) FOLLOWS FROM THE BINOMIAL THEOREM WITH $a = 2$ AND $b = 1$

$3^N = (2+1)^N = \sum_{K=0}^{N} \binom{N}{K} 2^K 1^{N-K} = \sum_{K=0}^{N} \binom{N}{K} 2^K$

(T) (h) 

(F) (i) FOR EXAMPLE, LET $P(x,y)$ MEAN "$x^2 \geq y$", THEN $\exists y \in \mathbb{R}$ S.T. $\forall x \in \mathbb{R}$, $x^2 \geq y$, NAMELY $y = 0$, BUT THERE IS NO $x \in \mathbb{R}$ S.T. $\forall y \in \mathbb{R}$, $x^2 \geq y$, BECAUSE THEN FOR INSTANCE $x^2 \geq 1$ AND $x^2 \geq 2$ AND $x^2 \geq 3$ ETC., SO $x = \pm\infty$ WHICH IS NOT IN $\mathbb{R}$.

(F) (j) FOR INSTANCE $f(N) = 1$ IS NOT ONTO!

2. *(10 points)* Determine whether or not the following two propositions are logically equivalent:

$$① = (p \Rightarrow q) \wedge ((\sim p) \Rightarrow r)$$

$$② = (p \Rightarrow r) \wedge ((\sim p) \Rightarrow q)$$

LET'S DO IT WITH TRUTH TABLES (BUT ANY OTHER METHOD IS OK TOO)

p	q	r	$p \Rightarrow q$	$\sim p \Rightarrow r$	①	$p \Rightarrow r$	$\sim p \Rightarrow q$	②
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F
T	F	T	F	T	F	T	T	T
T	F	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T	T
F	F	T	T	T	T	T	F	F
F	F	F	T	F	F	T	F	F

NOTICE THAT THE TWO TRUTH TABLES DO NOT COINCIDE, HENCE THE EXPRESSIONS ARE NOT LOGICALLY EQUIVALENT!

NOTE: NOTICE THAT IT'S ACTUALLY ENOUGH JUST TO SHOW THAT THE SECOND LINES OF THE TRUTH TABLES DO NOT COINCIDE!

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3. *(10 points)* Show that there is no rational number r such that

$$r^3 + r = 1$$

Hint: The proof is similar to the proof that $\sqrt{2}$ is irrational. You may need to argue in cases, depending on whether things are even or odd. You are allowed to use (without proof) all the facts that we've learned about powers/products of odd/even integers.

PROOF BY CONTRADICTION

SUPPOSE THERE IS A RATIONAL NUMBER $r = \frac{a}{b}$ SUCH THAT

$$r^3 + r = 1$$

WITHOUT LOSS OF GENERALITY, ASSUME a & b HAVE NO FACTORS IN COMMON.

PLUGGING IN $r = \frac{a}{b}$ INTO OUR EQUATION, WE GET

$$\left(\frac{a}{b}\right)^3 + \left(\frac{a}{b}\right) = 1$$

$$\frac{a^3}{b^3} + \frac{a}{b} = 1$$

$$\frac{a^3 + ab^2}{b^3} = 1$$

$$a^3 + ab^2 = b^3$$

NOW WE NEED TO ARGUE IN CASES:

CASE 1 a EVEN, b ODD,

BUT THEN $a^3 + ab^2 = b^3$, SO WE GET EVEN = ODD $\Rightarrow\Leftarrow$

(EVEN)^3 = EVEN (EVEN)(ODD)^2 = EVEN (ODD)^3 = ODD

CASE 2 a ODD, b EVEN

BUT THEN $a^3 + ab^2 = b^3$

ODD ODD×EVEN EVEN

ODD + EVEN = EVEN

(SEE BACK-PAGE)

ODD = EVEN $\Rightarrow\Leftarrow$

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CASE 3 a ODD, b ODD

BUT THEN $a^3 + ab^2 = b^3$

ODD + ODD = ODD

EVEN = ODD $\Rightarrow\Leftarrow$

CASE 4 a EVEN, b EVEN

BUT THEN a AND b HAVE A FACTOR IN COMMON (NAMELY 2) $\Rightarrow\Leftarrow$

IN ALL CASES, WE DERIVE A CONTRADICTION $\Rightarrow\Leftarrow$

4. *(10 points)* Recall that the Fibonacci numbers are defined by

$$F_0 = 0, F_1 = 1, F_{n+2} = F_n + F_{n+1}$$

Show by induction that F_n must satisfy the following identity:

$$F_0F_1 + \cdots + F_{2n-1}F_{2n} = (F_{2n})^2$$

Note: Just to clarify, the sum on the left-hand-side is $\sum_{i=0}^{2n-1} F_iF_{i+1}$

PROOF BY INDUCTION

LET P_N BE THE PROPOSITION THAT $F_0F_1 + \cdots + F_{2N-1}F_{2N} = (F_{2N})^2$

BASE CASE $N=1$

THEN $F_0F_1 + F_1F_2 = 0\times 1 + 1\times 1 = 1 = (F_2)^2$ ✓

INDUCTIVE STEP

SUPPOSE P_N IS TRUE, THAT IS

$F_0F_1 + \cdots + F_{2N-1}F_{2N} = (F_{2N})^2$

WANT TO SHOW P_{N+1} IS TRUE, THAT IS

$F_0F_1 + \cdots + F_{2(N+1)-1}F_{2(N+1)} = (F_{2(N+1)})^2$

THAT IS $F_0F_1 + \cdots + F_{2N+1}F_{2N+2} = (F_{2N+2})^2$

BUT $F_0F_1 + \cdots + F_{2N+1}F_{2N+2}$

$= \underbrace{F_0F_1 + \cdots + F_{2N-1}F_{2N}} + F_{2N}F_{2N+1} + F_{2N+1}F_{2N+2}$ INDUCTIVE HYPOTHESIS

$= (F_{2N})^2 + F_{2N}F_{2N+1} + F_{2N+1}F_{2N+2}$ BY DEF OF F_N WITH $2N$ INSTEAD OF N

$= (F_{2N})^2 + F_{2N}F_{2N+1} + F_{2N+1}(F_{2N} + F_{2N+1})$

$= (F_{2N})^2 + F_{2N}F_{2N+1} + F_{2N+1}F_{2N} + (F_{2N+1})^2$

$= (F_{2N})^2 + 2F_{2N}F_{2N+1} + F_{2N+1})^2$

$= (F_{2N} + F_{2N+1})^2 = (F_{2N+2})^2$ (BY DEF OF F_N) ✓

HENCE P_{N+1} IS TRUE, AND HENCE P_N IS TRUE $\forall N$, THAT IS $F_0F_1 + \cdots + F_{2N-1}F_{2N} = (F_{2N})^2$

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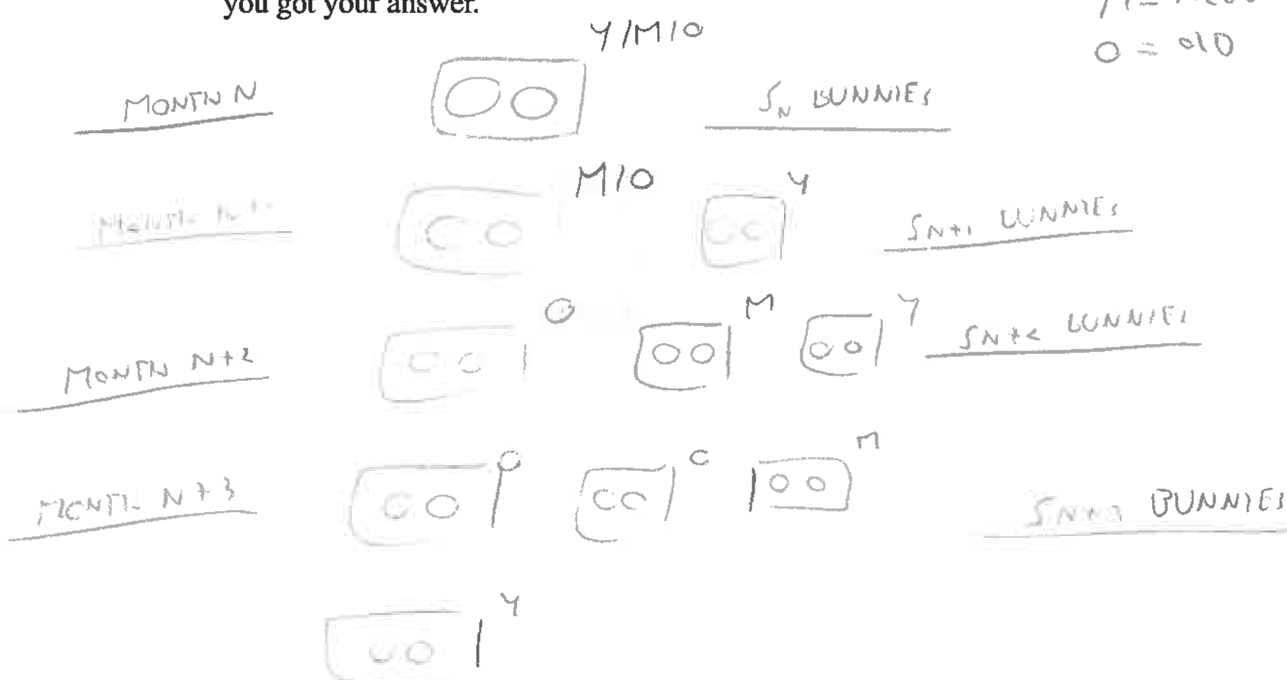
5. *(10 points)* Welcome to Bun-galore, a city purely inhabited by bunnies! A single pair of bunnies (male and female) is born at the beginning of a year. Assume the following conditions hold:

   (1) No bunnies ever die.

   (2) Bunnies are not fertile during their first two months of their life, but thereafter give birth to three new male/female pairs at the end of every month.

   Let s_n be the number of bunny pairs at the end of month n, with $s_0 = 1$. Find a recurrence relation for s_n, and briefly explain how you got your answer.

Y = YOUNG
M = MEDIUM
O = OLD



S_{N+3} = BUNNY PAIRS ALIVE @ THE END OF MONTH N+3

= BUNNIES ALIVE @ THE END ON MONTH N+2
+ BUNNY BORN IN MONTH N+3

= S_{N+2} + 3(# OLD BUNNY PAIRS IN MONTH N+2)

= S_{N+2} + 3(# BUNNIES ALIVE IN MONTH N) (B/C TAKES 2 MONTHS TO MATURE)

= $S_{N+2} + 3S_N$

$S_{N+3} = S_{N+2} + 3S_N$)

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6. *(10 points)* Oh noes!!! The Dark Lord Bun-ondorf stole Peyam's two fluffy bunnies Oreo and Cookie, and escaped to his evil fortress. You try to enter it, but unfortunately the main door is locked, but next to you, there is a well with the following inscription: "Hey, listen! If you throw in 15 diamonds in this well, then a fairy will appear who will solve all your problems. The order you throw in the diamonds doesn't matter, but you need to throw in at least 2 red diamonds and 1 blue diamond" Next to you, you find a bag with 20 Red diamonds, 20 Blue diamonds, 20 Yellow diamonds, 5 Green diamonds, and 5 Purple diamonds. How many different ways can you throw the diamonds in the well? You do **NOT** need to simplify your answer!

THIS IS LIKE THE DONUT PROBLEM FROM THE PRACTICE EXAM (BUT W/ A LITTLE TWIST)

SINCE YOU KNOW THAT THERE MUST BE ≥ 2 RED DIAMONDS AND ≥ 1 BLUE DIAMOND, THE QUESTION REDUCES TO THE FOLLOWING PB

"HOW MANY WAYS CAN YOU THROW IN $15-2-1=12$ DIAMONDS, IF YOU HAVE 18 RED DIAMONDS, 19 BLUE, 20 YELLOW, 5 GREEN, AND 5 PURPLE?" (LIMITED # OF DIAMONDS!)

HERE, WANT $|G_{\leq 5} \cap P_{\leq 5}|$

$= \text{TOTAL} - |G_{\geq 6} \cup P_{\geq 6}|$ (BY COMPLEMENT)

12 X, 4 BARS

$= \binom{16}{12} - |G_{\geq 6}| - |P_{\geq 6}| + |G_{\geq 6} \cap P_{\geq 6}|$ (INCLUSION–EXCLUSION)

$$|G_{\leq 5} \cap P_{\leq 5}| = \binom{16}{12} - \binom{10}{6} - \binom{10}{6} + \binom{4}{4} \ (=1)$$

6 X, 4 BARS — 6 X, 4 BARS — 1 X, 4 BARS

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∴) HENCE $P(\geq 2 \text{ ACES}) = 1 - \frac{\binom{48}{5}}{\binom{52}{5}} - \frac{4 \times \binom{48}{4}}{\binom{52}{5}}$

) SO $P(4 \text{ ACES} \mid \geq 2 \text{ ACES}) = \dfrac{\frac{48}{\binom{52}{5}}}{1 - \binom{48}{5}/\binom{52}{5} - 4\times\binom{48}{4}/\binom{52}{5}} = \dfrac{48}{\binom{52}{5} - \binom{48}{5} - 4\times\binom{48}{4}}$

7. *(10 points)* Recall that a poker hand consists of 5 cards chosen from 52 cards (= 13 ranks × 4 suits)

Suppose your poker hand contains at least two aces. What is the probability that it contains all four aces? You do **NOT** need to simplify your answer.

WANT $P(4 \text{ ACES} \mid \geq 2 \text{ ACES}) = \dfrac{P(4 \text{ ACES} \cap \geq 2 \text{ ACES})}{P(\geq 2 \text{ ACES})}$

$= \dfrac{P(4 \text{ ACES})}{P(\geq 2 \text{ ACES})}$

) $P(4 \text{ ACES})$: ARGUE IN TERMS OF THE FIRST 4 CARDS VS. LAST CARD

JUST 1 CARD — $52 - 4 = 48$ REMAINING CARDS — 48 FOUR OUTCOMES

TOTAL = $\binom{52}{5}$, SO $P(4 \text{ ACES}) = \dfrac{48}{\binom{52}{5}}$

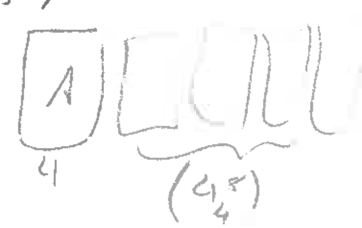
) TO FIND $P(\geq 2 \text{ ACES})$, USE COMPLEMENT RULE:

$P(\geq 2 \text{ ACES}) = 1 - P(< 2 \text{ ACES}) = 1 - P(\leq 1 \text{ ACE})$ ↙ ADDITION RULE

$= 1 - P(\text{NO ACE}) - P(1 \text{ ACE})$

SINCE THERE ARE 4 CARDS W/ ACES, $P(\text{NO ACE}) = \dfrac{\binom{48}{5}}{\binom{52}{5}}$ $\quad 52 - 4 = 48$

FOR $P(1 \text{ ACE})$, THERE ARE 4 CHOICES FOR FIRST CARD AND $\binom{48}{4}$ CHOICES FOR SECOND CARD,



SO $P(1 \text{ ACE}) = \dfrac{4 \times \binom{48}{4}}{\binom{52}{5}}$

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8. *(10 points)* A **full ternary tree** (from the Latin ternarius, meaning three at once) is a rooted tree for which every parent has exactly 3 children. Show that ~~a~~ full ternary tree of height h can only have at most 3^h leaves.

WE'LL DO STRONG INDUCTION ON h

LET P_h BE THE PROPOSITION THAT A FULL TERNARY TREE WITH HEIGHT h HAS AT MOST 3^h LEAVES.

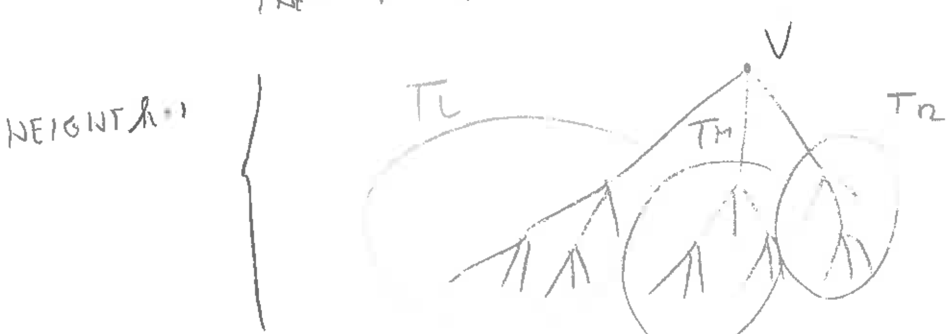
BASE CASE: $h = 0$ A FULL TERNARY TREE WITH HEIGHT 0 IS A VERTEX • WHICH HAS 1 LEAF (BY DEF OF LEAF, A VERTEX OF DEG = 0 IS A LEAF)

IND STEP: SUPPOSE P_k IS TRUE FOR $k = 0, 1, \ldots, h$, THAT IS A FULL TERNARY TREE W/ HEIGHT k HAS AT MOST 3^k LEAVES.

SHOW P_{h+1} IS TRUE, THAT IS A FULL TERNARY TREE W/ HEIGHT $h+1$ HAS AT MOST 3^{h+1} LEAVES.

LET T BE A FULL TERNARY TREE W/ HEIGHT $h+1$

THE T HAS A ROOT V.



CONSIDER THE LEFT SUBTREE T_L OF T. IT HAS HEIGHT $k \le (h+1) - 1 = h$ AND IS STILL A FULL TERNARY TREE, SO BY INDUCTION HYP, T_L HAS AT MOST $3^k \le 3^h$ LEAVES, AND THE SAME THING IS TRUE FOR THE MIDDLE SUBTREE T_M OF T AND THE RIGHT SUBTREE T_R OF T.

BUT THEN # OF LEAVES OF T = (# OF LEAVES OF T_L) + (# OF LEAVES OF T_M) + (# OF LEAVES OF T_R)

$$\le 3^h + 3^h + 3^h = 3 \times 3^h = 3^{h+1}$$

HENCE T HAS AT MOST 3^{h+1} LEAVES, HENCE P_{h+1} IS TRUE, AND BY IND, P_h IS TRUE $\forall h$.

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9. *(10 points)* **Disclaimer:** This problem is much harder than the rest. Please make sure to to thoroughly check your answers to the previous problems before tackling this problem.

Given a nonempty set A, show that the power set $\mathcal{P}(A)$ and the set $\mathcal{F}(A)$ of functions from A to $\{0, 1\}$ have the same cardinality.

Hint: It may be useful to consider the indicator function of a set, $\mathbb{1}_C$, defined by

$$\mathbb{1}_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

[redacted]

LET $f : \mathcal{P}(A) \to \mathcal{F}(A)$ BE DEFINED BY $f(C) = \mathbb{1}_C$ (INDICATOR FUNCTION OF C)

BEFORE WE CONTINUE, LET'S PROVE THE FOLLOWING CLAIM (WHICH WILL BE USEFUL LATER ON)

LEMMA: FOR ALL C, $(\mathbb{1}_C)^{-1}\{1\} = C$

PROOF: $[\subseteq]$ IF $x \in (\mathbb{1}_C)^{-1}\{1\}$, THEN $\mathbb{1}_C(x) = 1$ BY DEF OF INVERSE IMAGE. BUT IF $x \notin C$, THEN $1 = \mathbb{1}_C(x) = 0$, SO $1 = 0 \Rightarrow\Leftarrow$, HENCE $x \in C$ ✓

$[\supseteq]$ IF $x \in C$, THEN $\mathbb{1}_C(x) = 1$, SO $x \in \mathbb{1}_C^{-1}\{1\}$ ✓

WE MUST SHOW THAT f IS ONE-TO-ONE AND ONTO

ONE-TO-ONE: SUPPOSE $f(C) = f(D)$ (MUST SHOW $C = D$)

THEN $\mathbb{1}_C = \mathbb{1}_D$

SO $(\mathbb{1}_C)^{-1}\{1\} = (\mathbb{1}_D)^{-1}\{1\}$, BY LEMMA

SO $C = D$

ONTO: LET $g \in \mathcal{F}(A)$, THEN $g : A \to \{0,1\}$

LET $C = g^{-1}\{1\}$, AND WE NEED TO SHOW $g = f(C) = \mathbb{1}_C$ (SEE NEXT PAGE)

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BUT IF $x \in C = g^{-1}\{1\}$, THEN $g(x) = 1 = 1_C(x)$

AND IF $x \notin C = g^{-1}\{1\}$, THEN $g(x) \neq 1$, SO $g(x) = 0$ SINCE $g: A \to \{0,1\}$, AND SO $g(x) = 0 = 1_C(x)$

HENCE $g(x) = 1_C(x)$ $\forall x \in A$, AND SO $g = 1_C$ ∎