MATH 200 – FINAL EXAM

PEYAM RYAN TABRIZIAN

Name: ____

Instructions: Welcome to your Final Exam! You have 150 minutes (= 2 h 30) to take this exam, for a total of 100 points. **There are lots of problems, so do not spend more than 15 minutes on each problem.** Remember that you are not only graded on the correctness of your answer, but also on the clarity and completeness of your proofs. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. Good luck, and may the odds be in your favor! :)

Honor Code: I promise not to communicate with anyone about the exam unless everyone in my group has taken it, and I will not use any books or notes or cheat sheets or personal electronic devices (**including calculators**).

Signature: ____

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		+
Total		100

Date: Saturday, May 13 – Sunday, May 21, 2017.

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1. *(10 points)* Determine whether or whether the following reasoning is valid or not. Feel free to add more columns if that helps you.

$$(p \wedge q) \Rightarrow \sim r$$
$$p \vee (\sim q)$$
$$(\sim q) \Rightarrow p$$
$$\therefore\ p \vee r$$

p	q	r	$(p \wedge q) \Rightarrow \sim r$	$p \vee (\sim q)$	$(\sim q) \Rightarrow p$	$p \vee r$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

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2. *(10 points)* Prove that there are infinitely many prime numbers.

   Note: You are allowed to use **without proof** any facts that you know about integers and/or divisibility.

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3. *(10 points)* Show **using strong induction** that if $n \geq 12$, then n cents can be changed using a combination of 3 cent and 7 cent coins.

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4. *(10 points, 5 points each)* The two parts of this problem are independent.

   (a) Show that if A, B, C are sets and $B \subseteq (C - A)$, then $A \cap B = \emptyset$

   (b) Give an example of something that is not a set and (briefly) show that it's not a set.

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5. *(10 points)* In this problem, you may use **without proof** any facts and properties that you know about countable sets. Moreover, you will need to use the following result:

   Fundamental Theorem of Algebra: A polynomial of degree n has at most n zeros (where $n \geq 1$).

   A real number x is called **algebraic** if it is the zero of a polynomial of degree n with rational coefficients, that is if there is an integer $n \geq 1$ and rational numbers $a_0, a_1, \cdots, a_n$ with $a_n \neq 0$ such that

   $$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

   Show that the set Alg of algebraic numbers is countable.

   Hint: Write Alg as the countable union of countable sets.

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6. *(10 points)* In this problem you may use **without proof** any facts that you know about rational numbers.

Define the following relation $\sim$ on $\mathbb{R}^{\star}$ (= the nonzero real numbers):

$$x \sim y \text{ if and only if } \frac{x}{y} \text{ is rational}$$

(a) *(7 points)* Show that $\sim$ is an equivalence relation on $\mathbb{R}^{\star}$

(b) *(3 points)* Find $[1]$ (the equivalence class of 1)

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7. *(10 points)* The two parts of this problem are independent of each other, and you do **NOT** have to simplify your answer

   (a) *(5 points)* Urn 1 contains 10 red balls and 25 green balls, and Urn 2 contains 25 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a crooked coin with probability of 0.4 of landing heads. If the coin lands heads, the Urn 1 is chosen; otherwise Urn 2 is chosen. Then a ball is picked at random from the chosen urn. If the chosen ball is green, what is the probability that it was picked from Urn 1?

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(b) *(5 points)* Today, Café Peyam serves Apple, Blueberry, Chocolate, and Yam Pies. How many different selections of 20 pies contain at least 8 Apple Pies and at most 5 Blueberry Pies?

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8. *(10 points)*

(a) *(5 points)* Show that

$$\sum_{k=0}^{n} k \binom{n}{k} \left(\frac{1}{2}\right)^{n-1} = n$$

Hint: First use the binomial theorem to find an expansion of $\left(x + \frac{1}{2}\right)^n$. Then differentiate[1] both sides with respect to x, and finally let $x = \frac{1}{2}$.

[1]In case you don't know what differentiating is, it's an operation denoted by $'$ with the following rules:

(1) For any constants c_k and any functions f_k,

$$\left(\sum_{k=0}^{n} c_k f_k(x)\right)' = \sum_{k=0}^{n} c_k f_k'(x)$$

(2) For any $n \geq 0$ and $k \geq 0$,

$$\left(\left(x + \frac{1}{2}\right)^n\right)' = n\left(x + \frac{1}{2}\right)^{n-1} \qquad \left(x^k\right)' = kx^{k-1}$$

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(b) *(5 points)* Use your answer in (a) to find the expected number of heads when n (fair) coins are tossed.

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9. *(10 points, 2 points each)* Give an example (or show that one does not exist)

   (a) A graph with 4 vertices of degrees 1, 1, 1, 4

   (b) A graph of total degree 8 with 2 vertices.

   (c) A simple graph with 8 vertices and 8 edges.

   (d) A tree with 9 edges and 7 vertices.

   (e) A full binary tree of height 3 and total degree 22

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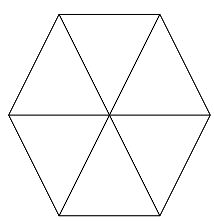
10. *(10 points, 2 points each)* Give an example (or show that one does not exist)

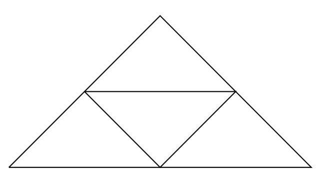
    (a) A full binary tree of height 3 and 9 terminal vertices (leaves)

    (b) A full binary tree of total degree 21 with 5 internal vertices (branches).

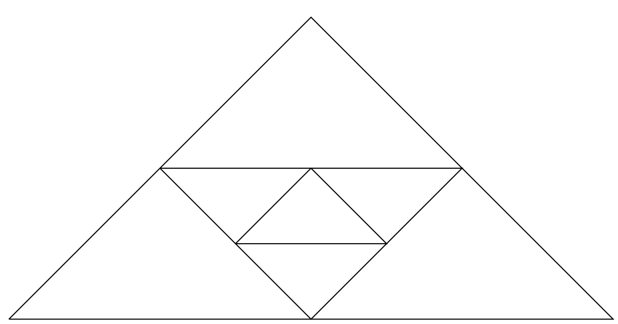
    (c) An Euler Circuit in the following graph:



    (d) An Euler Circuit in the triforce:



    (e) A Hamiltonian circuit in the mega-triforce:



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11. *(0 points)* **Note:** This problem is worth 0 points and is just meant for the people who are trying to get an A+ in this course. Please thoroughly review your answers to the other questions before tackling this. Also this is **NOT** a guarantee for an A+; at the end, it all depends on my own discretion.

    If A is set, show that there is a one-to-one function from A to $\mathcal{P}(A)$ but no onto function from A to $\mathcal{P}(A)$ (so, in some sense, $\mathcal{P}(A)$ is strictly larger in size than A).

    Hint: Give an argument à la Russel's Paradox.

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