MATH 453 – FINAL EXAM

PEYAM RYAN TABRIZIAN

Name: ____

Instructions: Welcome to your final exam! You have 24 hours to take this exam, for a total of 100 points. Write in full sentences whenever you can and try to be as precise as you can. If you need to continue your work on the back of a page, please indicate that you're doing so, or else your work may be discarded. May your luck be noncharacteristic! :)

Honor Code: I promise not to communicate with anyone about this exam unless everyone in my group has already taken it. I also promise not to use any books or notes or cheat sheets or personal electronic devices (including calculators).

Signature: ____

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Date: Saturday, May 13 – Sunday, May 21.

(Back-page)

1. *(10 points)* Prove uniqueness of solutions of Poisson's equation:

$$\begin{cases} -\Delta u = f & \text{in } W \\ u = g & \text{on } \partial W \end{cases}$$

(a) Using energy methods

(b) Using the (weak or strong) maximum principle

(Back-page)

2. *(10 points)* State and prove D'Alembert's formula for the solution of the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R} \times \{t = 0\} \\ u_t(x,0) = h(x) & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

Note: You may use **without proof** that the solutions of the nonhomogeneous transport equation

$$\begin{cases} u_t + b \cdot Du = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x,0) = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

are given by

$$u(x,t) = g(x - tb) + \int_0^t f(x + (s-t)b, s)ds$$

(Back-page)

(More space to work on 2)

(Back-page)

3. *(10 points)* Suppose that u is a minimizer of

$$I[w] = \int_W L(Dw(x), w(x), x)dx$$

among all smooth functions $w : \overline{W} \to \mathbb{R}$ satisfying the boundary condition $w = g$ on ∂W, where $L = L(p, z, x)$ and $g = g(x)$ are given.

What PDE must u satisfy? Explain how you got your answer.

(Back-page)

4. *(10 points)* Find a solution of the heat equation on the half-line:

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0,\infty)\times(0,\infty) \\ u(x,0) = g(x) & \text{on } (0,\infty)\times\{t=0\} \\ u_x(0,t) = 0 & \text{on } \{x=0\}\times(0,\infty) \end{cases}$$

Note: The fundamental solution for the heat equation in one dimension is $\Phi(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$

(Back-page)

(More space to work on 4)

(Back-page)

5. *(10 points)* Solve using characteristics:

$$\begin{cases} (u_{x_1})^3 + (u_{x_2})^3 = 2 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x_1, 0) = x_1 & \text{on } \mathbb{R} \times \{x_2 = 0\} \end{cases}$$

Note: The characteristic ODE are given by:

$$\begin{cases} \mathbf{x}'(s) = D_p F \\ z'(s) = \mathbf{p}(s) \cdot D_p F \\ \mathbf{p}'(s) = -D_x F - (F_z)\mathbf{p}(s) \end{cases}$$

(Back-page)

(More space to work on 5)

(Back-page)

6. Using separation of variables, find a solution of

$$\begin{cases} u_t = u_{xx} & 0 < x < 1, t > 0 \\ u_x(0,t) = 0, u_x(1,t) = 0 & t > 0 \\ u(x,0) = \cos(2\pi x) & 0 < x < 1 \end{cases}$$

(Back-page)

(More space to work on 6)

(Back-page)

7. *(10 points)* Let $u(x) = \frac{1}{|x|}$ on $B = B(0,1)$ (the open unit ball in $\mathbb{R}^n$), with $n \geq 5$.

   (a) Find the weak derivative Du of u and prove it is the weak derivative of u. Beware of the singularity at 0.
   (b) Is u in $H^1(B)$? Why or why not?

(Back-page)

(More space to work on 7)

(Back-page)

8. *(10 points)* Suppose that $u = u(x,t)$ is a smooth solution of

$$\begin{cases} u_t + \Delta u = 0 & \text{in } W \times [0,T] \\ u(x,t) = 0 & \text{on } \partial W \\ u(x,T) = h(x) & \text{on } W \times \{t = T\} \end{cases}$$

and that $v = v(x,t)$ is a smooth solution of

$$\begin{cases} v_t - \Delta v = 0 & \text{in } W \times [0,T] \\ v(x,t) = 0 & \text{on } \partial W \\ v(x,0) = 0 & \text{on } W \times \{t = 0\} \end{cases}$$

Use energy methods to show that

$$\int_W h(x)v(x,T)dx = 0$$

(Back-page)

9. *(10 points)*

(a) Prove the following variant of Cauchy's inequality (called Cauchy's inequality with ϵ):

For any $a, b \in \mathbb{R}$, and any $\epsilon > 0$, we have

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

Hint: Write $ab = (\sqrt{2\epsilon}a)\left(\frac{b}{\sqrt{2\epsilon}}\right)$ and use the standard Cauchy's inequality.

(b) Use (a) and Poincaré's inequality to show that if $u \in C_c^\infty(W)$ is a solution of Poisson's equation

$$\begin{cases} -\Delta u = f & \text{in } W \\ u = 0 & \text{on } \partial W \end{cases}$$

then there is a constant $C > 0$ depending *only* on W and the dimension n such that

$$\int_W (u(x))^2 dx \leq C \int_W (f(x))^2 dx$$

This shows that the 'energy' of u is uniformly bounded by the energy of f.

(Back-page)

(More space to work on 9)

(Back-page)

10. *(10 points)* Suppose that u solves $\Delta u = 0$ on $B(0, R)$ (for some fixed $R > 0$) and $u(0) = 0$. Our goal in this problem is to figure out how fast u goes to 0 at $x = 0$.

For $0 < r < R$, define the functions

$$a(r) := \frac{1}{r^{n-1}} \int_{\partial B(0,r)} |u(y)|^2 \, dS(y)$$
$$b(r) := \frac{1}{r^{n-2}} \int_{B(0,r)} |Du(x)|^2 \, dx$$

We define the *radial* derivative of u to be $u_r(x) := Du \cdot \frac{x}{|x|}$.

(a) Show that the following identities hold:

$$b(r) = \frac{1}{r^{n-2}} \int_{\partial B(0,r)} u(y) u_r(y) dS(y)$$
$$a'(r) = \frac{2}{r^{n-1}} \int_{\partial B(0,r)} u(y) u_r(y) dS(y)$$

Hint: For the first, use the definition of b and integrate by parts.

One can also show that (do **NOT** do this; it uses what's called Noether's theorem):

$$b'(r) = \frac{2}{r^{n-2}} \int_{\partial B(0,r)} (u_r(y))^2 dS(y)$$

(b) Deduce from (a) that $(b(r))^2 \leq \frac{r}{2} a(r) b'(r)$

(c) Define $f(r) := \frac{b(r)}{a(r)}$ (the frequency function) and deduce from (a) and (b) that $f'(r) \geq 0$. This is called Almgren's monotinicity formula (who was Frank Morgan's PhD advisor!)

(d) Deduce from (b) and (c) that $\frac{a'(r)}{a(r)} \leq \frac{\beta}{r}$ and therefore that $a(r) \geq \gamma r^{\beta}$, where $\beta := 2\frac{b(R)}{a(R)}, \gamma := \frac{a(R)}{R^{\beta}}$. Conclude by finding a constant C (in terms of γ) and a real number $s > 0$ (in terms of n and β) such that (yes, the whole ball)

$$\int_{B(0,r)} (u(x))^2 \, dx \geq C r^s$$

So, in terms of energies, u goes to 0 at rate at least r^s.

(Back-page)

(Space to work on 10)

(Back-page)

(More space to work on 10)

(Back-page)

(Even more space to work on 10)

(Back-page)

(Scratch paper)

(Back-page)