MATH 200 − FINAL EXAM

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Name: ____

Instructions: Welcome to your Final Exam! You have 150 minutes (= 2h30) to take this exam, for a total of 100 points. **Do not open the exam until you're instructed to do so**. Remember that you are not only graded on the correctness of your answer, but also on the clarity and completeness of your proofs. Write in full sentences whenever you can. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. Good luck, and may the odds be in your favor! :)

Honor Code: I promise not to communicate with anyone during the exam, and I will not use any books or notes or cheat sheets or personal electronic devices (**including calculators**).

Signature: ____

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total		100

Date: Thursday, December 15, 2016.

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1. *(20 points, 2 pts each)*

Label the following statements as **TRUE (T)** or **FALSE (F)**. Any correct answer gives you 2 points, and any incorrect answer gives you 0 points. You do **NOT** get negative points for an incorrect answer, and you do **NOT** need to justify your answers.

_____ (a) The probability that the answer to exactly 2 of the questions in this problem is T is $\frac{2}{10}$

_____ (b) $A \times B$ always has the same cardinality as $B \times A$, even if A and B don't necessarily have the same cardinality.

_____ (c) If a graph G has 4 vertices, each with degree $1, 2, 3, 4$, then G has an Euler circuit.

_____ (d) If events A and B are independent, then A^c and B^c are independent as well.

_____ (e) There is a full binary tree with 4 leaves and 6 branches.

_____ (f) If a, b, c are positive integers and a divides bc, then a divides b or a divides c.

_____ (g) The following identity holds for all $n \in \mathbb{N}$:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

_____ (h) There exists a graph with 4 vertices of degrees $0, 1, 2, 3$ respectively

_____ (i) If $\exists y \forall x P(x, y)$, then $\exists x \forall y P(x, y)$

_____ (j) Any function from $\{1, 2, 3, 4, 5\}$ to $\{1, 2\}$ must be onto.

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2. *(10 points)* Determine whether or not the following two statements are logically equivalent:

$$(p \Rightarrow q) \wedge ((\sim p) \Rightarrow r)$$

$$(p \Rightarrow r) \wedge ((\sim p) \Rightarrow q)$$

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3. *(10 points)* Show that there is no rational number r such that

$$r^3 + r = 1$$

Hint: The beginning of the proof is similar to the proof that $\sqrt{2}$ is irrational. You'll need to argue in cases, depending on whether things are even or odd. You are allowed to use (without proof) all the facts that we've learned about powers and products of even/odd integers.

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4. *(10 points)* Recall that the Fibonacci numbers are defined by

$$\begin{cases} F_{n+2} = F_n + F_{n+1} \\ F_0 = 0 \\ F_1 = 1 \end{cases}$$

Show by induction that F_n must satisfy the following identity:

$$F_0 F_1 + \cdots + F_{2n-1} F_{2n} = (F_{2n})^2$$

Note: Just to clarify, the sum on the left-hand-side is $\sum_{i=0}^{2n-1} F_i F_{i+1}$

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5. *(10 points)* Welcome to Bun-galore, a city purely inhabited by bunnies! A single pair of bunnies (male and female) is born at the beginning of the year. Assume the following conditions hold:

(1) No bunnies ever die.

(2) Bunnies are not fertile during their first two months of their life, but thereafter give birth to three new male/female pairs at the end of every month.

Let s_n be the number of bunny pairs at the end of month n, with $s_0 = 1$. Find a recurrence relation for s_n, and explain how you got your answer.

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6. *(10 points)* Oh noes!!! The Dark Lord Bun-ondorf stole Peyam's two fluffy bunnies Oreo and Cookie, and escaped to his evil fortress. You try to enter it, but unfortunately the main door is locked. Next to you, you find a well with the following inscription: "Hey, listen! If you throw in 15 diamonds in this well, then a fairy will appear and will open the door for you. The order you throw in the diamonds doesn't matter, but you need to throw in at least 2 red diamonds and at least 1 blue diamond" Next to you, you also find a bag with 20 Red diamonds, 20 Blue diamonds, 20 Yellow diamonds, 5 Green diamonds, and 5 Purple diamonds. How many different ways can you throw the diamonds in the well? You do **NOT** need to simplify your answer!

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7. *(10 points)* Recall that a poker hand consists of 5 cards chosen from 52 cards (= 13 ranks × 4 suits)

   Suppose your poker hand contains at least two aces. What is the probability that it contains all four aces? You do **NOT** need to simplify your answer.

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8. *(10 points)* A **full ternary tree** (from the Latin ternarius = three) is a rooted tree for which every parent has exactly 3 children. Show that a full ternary tree of height h has at most 3^h leaves.

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9. *(10 points)* **Disclaimer:** This problem is much harder than the rest. Please make sure to to thoroughly check your answers to the previous problems before tackling this problem.

Given a nonempty set A, show that the power set $\mathcal{P}(A)$ of A and the set $\mathcal{F}(A)$ of functions from A to $\{0, 1\}$ have the same cardinality.

Hint: It may be useful to consider the indicator function of a set C, 1_C, defined by

$$1_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

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