MATH 379 – MIDTERM

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Name: ____

Instructions: Welcome to your midterm! You have 50 minutes to take this exam, for a total of 100 points. Write in full sentences whenever you can and try to be as precise as you can. If you need to continue your work on the back of a page, please indicate that you're doing so, or else your work may be discarded. Good luck, and may the Ansatz be with you! :)

Honor Code: I promise not to communicate or collaborate with anyone during the exam, and I will not use any books or notes or cheat sheets or personal electronic devices (including calculators).

Signature: ____

1		50
2		30
3		20
Total		100

Date: Friday, October 28, 2016.

1. *(50 points)*

Consider the following equation, where $u_\epsilon = u_\epsilon(t)$:

$$u_\epsilon'' - \epsilon \cos(t) \sin(u_\epsilon) = 0$$

Let our Ansatz be:

$$u_\epsilon = u_0(t, \epsilon t) + \epsilon\, u_1(t, \epsilon t) + \cdots$$

where $u_k = u_k(t, \tau)$ and $t \mapsto u_k(t, \tau)$ is 2π periodic.

(a) Use the $O(1)$ terms to show that u_0 does not depend on t, that is $u_0 = u_0(\tau)$.
(b) Use the $O(\epsilon)$ terms to show that $u_1 = -\sin(u_0(\tau))\cos(t)$ (Set any constants you may find to 0)
(c) Use the $O(\epsilon^2)$ terms to conclude $u^0_{\tau\tau} + \frac{1}{2}\sin(u_0)\cos(u_0) = 0$
Hint: Use $\int_0^{2\pi} \cos^2(t) dt = \pi$

2. *(30 points)* Laplace's method states that if φ is a smooth function that has a global max at x_0 with $\varphi'(x_0) = 0$ and $\varphi''(x_0) < 0$, and a is any smooth function (not necessarily with compact support), then, as $\epsilon \to 0$,

$$\int_0^\infty a(x) e^{\frac{\varphi(x)}{\epsilon}} dx = \sqrt{\frac{2\pi\epsilon}{|\varphi''(x_0)|}} e^{\frac{\varphi(x_0)}{\epsilon}} a(x_0) + o(\sqrt{\epsilon}).$$

Use Laplace's method to show that, as $n \to \infty$,

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} + o\left(n^{n+\frac{1}{2}}\right)$$

Make sure to specify what φ, x_0, a and ϵ you're using.

Hint: Start with the identity (you don't have to prove this)

$$(n-1)! = \int_0^\infty e^{-t} t^{n-1} dt$$

and use the substitution $s = \frac{t}{n}$. Make sure to keep a $\frac{1}{s} ds$ in your integral!

3. *(20 points)* [Harder]

Using **the definition** of asymptotic expansions, show that if

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n \quad \text{and} \quad g(\epsilon) \sim \sum_{n=0}^{\infty} b_n \epsilon^n$$

and both f, g and their asymptotic expansions are bounded, then

$$f(\epsilon)g(\epsilon) \sim \sum_{n=0}^{\infty} c_n \epsilon^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Hint: The following formula might come in handy:

$$\left(\sum_{k=0}^{n} x_k\right)\left(\sum_{k=0}^{n} y_k\right) = \sum_{m=0}^{n} \sum_{k=0}^{m} x_k y_{m-k}$$

Note: Strictly speaking, this hint is incorrect in general, but it is correct for this problem; see the solutions!