MATH 379 – MIDTERM – STUDY GUIDE

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The Math 379 – Midterm will take place on **Friday, October 28, 2016**, from 10 AM to 10:50 AM in 6 Griffin Hall (our usual lecture-room). It covers everything that we've learned, up to and including Lecture **16** (Example 3: WKB Method, Wednesday, October 19). It will be a closed book and closed notes-exam, and no cheat-sheets will be allowed. That said, don't worry, you are certainly not expected to memorize any formulas (cough, cough, Laplace's method), unless I explicitly tell you otherwise below.

This is a study guide for the exam, and hopefully covers everything you need to know for the exam.

Main concepts: Multiple Scales (Chapters 1 and 3), Laplace's Method/Stationary Phase (Chapter 2)

1. Chapter 1: Introduction

1.1. Example 1: Acoustic Approximation in Fluid Mechanics.

- Make sure you are comfortable with all the notation I have introduced, like $Du, \nabla u, \Delta u, D^2 u$, as well as the notation for vector-values functions, like $D\mathbf{u}$, $div(\mathbf{u})$. In case you're still confused about it, refer to the notation-handout I had in class.
- This is a complicated example, but hopefully it's easier now that you're more comfortable with multiple scales. The most important thing I'd like you to know how to do is to do an Ansatz and plug it into an equation, and compare term by term. You are not expected to figure out exactly what equations our ρ^1 satisfies, but I *could* ask you: Show that ρ^1 solves equation X.
- Know the notation $f = o(\epsilon)$ and $f = O(\epsilon)$ and its variants like $f = o(\epsilon^2)$ etc.
- Know the fact that says that if $a_0 + a_1\epsilon + \cdots = b_0 + b_1\epsilon + \cdots$, then for all i, $a_i = b_i$.

Date: Friday, October 28, 2016.

1.2. **Example 2: Perturbation of eigenvalues.**

- You can certainly ignore this section, because it involves a bunch of linear algebra you're not responsible for. But again, know at least how to do the Ansatz and what you get when you compare the $O(1)$ and $O(\epsilon)$ terms.

1.3. **Example 3: Derivation of the KdV equation.**

- As mentioned many many times, this is a very ridiculous problem, and you really don't need to understand the details of it (especially since there's this one very weird trick at the end)
- However, I would say you do need to know how to simplify your equations after you do the change of variables (see Problem 1 on HW 2), how to do the Ansatz, and figure out what to conclude what equations you get **at least** for (A), (B), and (C), but **don't** worry what happens afterwards for (D) and how to combine them to get a KdV equation
- As far as the section on the theory of the KdV equation goes, you do **not** need to know what the general KdV equation looks like, and you do **not** need to know what the form of the solutions looks like. **BUT** once I tell you what they look like, you **DO** need to know how to plug them in, andyou **DO** need to know that trick of multiplying your equation by φ'.

1.4. **Theoretical Aspects.**

- **Know the definition of an asymptotic expansion, that is know what $f(\epsilon) \sim \sum_{k=0}^{\infty} a_k \epsilon^k$ means**
- Know how to prove the lemma about uniqueness of asymptotic expansions
- Know how to show that $e^{-\frac{1}{\epsilon}}$ has asymptotic expansion 0
- In Lecture 6 (Wednesday, September 21), I have proved another lemma about constructing a function with a given asymptotic expansion. You don't need to memorize the proof, but make sure to understand at least the main steps. In particular, make sure to know what a support function is.
- Know the more general definition of asymptotic expansion i.e. $f \sim \sum_{k=0}^{\infty} a_k \phi_k(\epsilon)$.

2. Chapter 2: Asymptotic Evaluation of Integrals

Disclaimer: You do NOT need to know the formulas for Laplace's method and Stationary phase, I will provide them for you if necessary!

2.1. Laplace's Method.

- Know what the assumptions on a and φ of Laplace's method are.
- Know what $I[\epsilon]$ is
- You do **NOT** need to know the (crazy) proof of Laplace's method in the special case, you can skip it if you like.
- Know the statement of Morse Lemma
- You do not need to memorize the proof of the general Laplace's method, but at least know the general ideas.
- That said, at least know the outline behind what we're doing with Laplace's method, that is you start with the special case, and then use Morse Lemma to reduce the general case to the special case.
- Know how to calculate the first couple of terms in the general Laplace method, although I won't go too crazy :) In particular, know what η and ψ represent.
- Know the formula $\int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$ (you can always reprove it in case you don't remember)
- Know what to do in case $\varphi(0) \neq 0$ or the maximum is at x_0 instead of 0
- Know what the multi-index notation is, and memorize Taylor's forula in several variables
- Again, know everything above, but in several dimensions

2.2. Stationary Phase.

- For stationary phase, again you're not responsible for the formulas above, and you do not need to know the proof about rapid decay or how to prove the special case.
- You are **NOT** responsible for knowing things about the Fourier transform, this is not Applied Real Analysis after all :)
- And again, you do not need to memorize the formulas for stationary phase, but know how to use them
- Know what the sign of a matrix is (number of positive eigenvalues minus the number of negative eigenvalues)

2.3. Application: Group vs. Phase Velocity.

- Skip the section on group vs. phase velocity, that section is kinda weird, to be honest

3. Chapter 3: Multiple Scales

This is hopefully much more in tune with what you need to know how to do :)

3.1. Example 1: Rapidly Oscillating Coefficients.

- Know how to apply your usual Ansatz to this equation; you don't need to know why the Ansatz doesn't work
- Know how to apply your better Ansatz to this equation, and find the $O(\frac{1}{\epsilon^2})$, $O(\frac{1}{\epsilon})$ and $O(1)$ terms. I will always give you the Ansatz that you need to plug in.
- Know the trick of multiplying by u^0 and then showing that $u^0_y = 0$
- For the $O(\frac{1}{\epsilon})$-term, know how to rewrite your equation in divergence form
- You **don't** need to know the trick of introducing th efunction w, but do know how to use it
- For the $O(1)$-term, know the trick of just integrating the equation and pulling out the terms that don't depend on y.

3.2. Example 2: An oscillator with damping.

- You don't need to know how to find the conserved quantity $g(t)$, but you do need to know how to show that it's conserved, and you do need to know how to deduce that u^ϵ is bounded from that.
- Again, know how to do the Ansatz
- Know how to solve the ODEs in the $O(\epsilon)-$term (like on your homework). You are responsible for undetermined coefficients, but not for variation of parameters.
- Know how to plug in the better Ansatz into your equation. Again, I will tell you which Ansatz to plug in, and know how to obtain the given equations for A and B.
- Know how we obtained the equations for A' and B'. You don't need to know how to write $\cos^3$ in terms of $\cos$ and $\sin$, but understand that we selected A and B to kill the resonance terms.
- You don't need to memorize Hamilton's equations; I will give them to you if necessary, but do know how to prove the lemma that $H(x(t), p(t))$ is a conserved quantity, and do know how to put an ODE in Hamiltonian form (all you need to do is to antidifferentiate H_x and H_p). Also, understand how to show that $|A|$ and $|B|$ are bounded from the fact that the Hamiltonian is constant.

3.3. Example 3: WKB Method.

- You don't need to understand why our first Ansatz didn't work, but you do need to know what to get when you plug in your Ansatz (don't worry about the part where I said skip the algebra)
- Know how to plug in the better Ansatz. I *will* give you the requirements on σ^{ϵ}, you don't need to figure them out, but you do need to figure out why we chose σ^{ϵ} to have the form that we want.
- Everything else in this example is fair game, especially how solve for A and B and how to figure out what u^{ϵ} looks like. You don't need to worry about the change-of-variables into θ-part

4. HOMEWORK-PROBLEMS

The Homework Problems are a great practice for the exam questions, try to do them without the hints if possible!

- **Homework 1:** Don't worry about Problem 1, but in Problem 2, do know how to do (a) (without the hint!), (b) (again, without any help, except that I will give you what u_k is), and (c) and (d) without the hint.
- **Homework 2:** Don't worry about Problem 1 (but know how to do it), **definitely** know how to do Problem 2 without the hints! (except in (c) I would tell you what the equation is)
- **Homework 3:** Problem 1 is also an excellent problem. In that problem, I would give you the equation and the form of u, but I wouldn't give you any hints, except I would tell you the substitution. Problem 2 is also a great Laplace method question. A good exam problem would be to do (b)-(d) without the hints, except that I would give you the statement of Laplace's method, as well as the formula in (a). Know the definition of $f(n) \sim g(n)$ as $n \to \infty$.
- **Homework 4:** Obviously I won't give you anything as ridiculous as Problem 1, except that I could ask you how to do it in a really special case, and I could ask you about how to calculate $L_0 a$ (but I'd give you all the formulas that you need, except I won't define η or ψ). And I would provide you with formulas for C_2 if necessary. Problem 2 is also a great exam problem. Try to do it without the hint of second-order Taylor expansion and without the hint of the integral of $e^{-\frac{x^2}{2}}$.
- **Homework 5:** Also good problems, although Problem 2 is a bit inappropriate for the exam (this is not a calculus-course), but know how to do undetermined coefficients and how to solve second-order constant coefficient differential equations.
- **Homework 6:** Both problems are **excellent** exam problems. In Problem 1, know how to do the Ansatz and know why we chose

A and B to kill the resonance terms, and know how to find the exact solution of the differential equation. For Problem 2, know how to plug in the Ansatz, know the trick of Taylor-expanding out $\sin(\theta^0 + \epsilon\theta^1)$ (I might not tell you about it!) Also know the trick of multiplying by a function and integrating by parts, and know how to do part (b). You don't need to know the formula for the integral of $\cos^2$.