MIDTERM - SOLUTIONS

PROBLEM 1

(a) $U^{\varepsilon} = U^{0}(t, \varepsilon t) + \varepsilon U^{1}(t, \varepsilon t) + \dots$

$$U_K' = U_t^K + \varepsilon U_\tau^K$$

$$U_K'' = U_{tt}^K + 2\varepsilon U_{t\tau}^K + \varepsilon^2 U_{\tau\tau}^K$$

PLUGGING INTO OUR EQUATION, WE GET

$$U_\varepsilon'' - \varepsilon \cos(t) \sin(U_\varepsilon) = 0$$

$$U_0'' + \varepsilon U_1'' - \varepsilon \cos(t) \sin(U_0 + \varepsilon U_1) = 0$$

$$U_{tt}^0 + 2\varepsilon U_{t\tau}^0 + \varepsilon^2 U_{\tau\tau}^0 + \varepsilon U_{tt}^1 + 2\varepsilon^2 U_{t\tau}^1 + \varepsilon^3 U_{\tau\tau}^1$$
$$- \varepsilon \cos(t) \sin(U_0) - \varepsilon^2 U^1 \cos(U_0) \cos(t) = 0$$

↳ TAYLOR EXP $\sin(\sigma^0 + \varepsilon\sigma^1) = \sin(\sigma^0) + \varepsilon \sigma^1 \cos(\sigma^0)$

O(1) - TERMS $\quad U_{tt}^0 = 0$

MULTIPLY BY U^0 AND $\int$ OVER $[0, 2\pi]$:

$$\int_0^{2\pi} U_{tt}^0 U^0 \, dt = 0$$

NO BDY TERMS SINCE $t \mapsto U_t^0$ IS PERIODIC

$$- \int_0^{2\pi} \underbrace{(U_t^0)^2}_{\geq 0} dt = 0$$

$\Rightarrow U_t^0 = 0$

$\Rightarrow U^0 = U^0(\tau)$

(b) $\underline{O(\varepsilon)\text{-TERMS}}$ $$2\cancel{U^0_{t\tau}}^{\,0} + U^1_{tt} - \cos(t)\sin(U_0) = 0$$
$$(U^0 = U^0(\tau))$$

$$U^1_{tt} = \cos(t)\sin(U_0)$$

$$\Rightarrow \quad U^1_t = \sin(t)\sin(U_0) + \overset{\text{SET} = 0}{A}$$

$$\Rightarrow \quad U^1(t) = -\cos(t)\sin(U_0) + \overset{\text{SET} = 0}{B}$$

$$\Rightarrow \quad U^1(t) = -\cos(t)\sin(U^0)$$

(c) $\underline{O(\varepsilon^2)\text{-TERMS}}$ $$U^0_{\tau\tau} + 2U^1_{t\tau} - U^1\cos(U_0)\cos(t)$$

$$U^0_{\tau\tau} + 2\left(-\cos(t)\sin(U^0(\tau))\right)_{t\tau} + \cos^2(t)\sin(U_0)\cos(U_0)$$

$$U^0_{\tau\tau} + 2\sin(t)\cos(U^0)U^0_\tau + \cos^2(t)\sin(U_0)\cos(U_0) = 0$$

INTEGRATE W.R.T. t ON $[0, 2\pi]$

$$\int_0^{2\pi} U^0_{\tau\tau} + 2\sin(t)\cos(U^0)U^0_\tau + \cos^2(t)\sin(U_0)\cos(U_0)\,dt = 0$$

$$U^0_{\tau\tau}(2\pi) + 2\cos(U^0)U^0_\tau \underbrace{\int_0^{2\pi}\sin(t)\,dt}_{0} + \sin(U_0)\cos(U_0)\underbrace{\int_0^{2\pi}\cos^2(t)\,dt}_{\pi} = 0$$

$$2\pi U^0_{\tau\tau} + \pi\sin(U_0)\cos(U_0) = 0$$

$$U^0_{\tau\tau} + \frac{1}{2}\sin(U_0)\cos(U_0) = 0$$

$$\left(U^0_{\tau\tau} + \frac{1}{4}\sin(2U_0) = 0 \right)$$

PROBLEM 2

$$(N-1)! = \int_0^\infty e^{-t} t^{N-1}\, dt$$

CHANGE OF VARS $s = \frac{t}{N}$

$dt = N\,ds$

$$= \int_0^\infty e^{-Ns} (Ns)^{N-1} N\, ds$$

$$= N^N \int_0^\infty e^{-Ns} s^N \frac{1}{s}\, ds$$

$$= N^N \int_0^\infty e^{-Ns} e^{N\ln(s)} \frac{1}{s}\, ds$$

$N = \frac{1}{\varepsilon}$

$$= N^N \int_0^\infty e^{\frac{-s+\ln(s)}{\varepsilon}} \frac{1}{s}\, ds$$

NOW APPLY LAPLACE WITH $a(s) = \frac{1}{s}$, $\varphi(s) = -s + \ln(s)$

NOTICE THAT $\varphi'(s) = -1 + \frac{1}{s} = 0 \implies s = 1$

AND $\varphi''(s) = -\frac{1}{s^2}$, SO $\varphi''(1) = -1 < 0$

SO IF YOU SET $x_0 = 1$ THEN φ HAS A GLOBAL MAX @ 1, AND

SO BY LAPLACE

$$\int_0^\infty e^{\frac{-s+\ln(s)}{\varepsilon}} \frac{1}{s}\, ds = \sqrt{\frac{2\pi\varepsilon}{|\varphi''(1)|}}\, e^{\frac{\varphi(1)}{\varepsilon}} \frac{1}{1} + o(\sqrt{\varepsilon})$$

$$= \sqrt{\frac{2\pi}{N}}\, e^{-N} + o\left(\sqrt{\frac{1}{N}}\right)$$

AND SO $$(N-1)! = N^N \left(\sqrt{\frac{2\pi}{N}}\, e^{-N} + o\left(\sqrt{\frac{1}{N}}\right)\right)$$

$$= \left(N^{N-\frac{1}{2}} \sqrt{2\pi}\, e^{-N}\right) + o\left(N^{N-\frac{1}{2}}\right)$$

SO MULTIPLYING BOTH SIDES BY N, WE GET

$$N \times (N-1)! = N \times N^{N-\frac{1}{2}} \sqrt{2\pi}\, e^{-N} + N \times o(N^{N-\frac{1}{2}})$$

$$N! = \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N} + o(N^{N+\frac{1}{2}})$$

PROBLEM 3

LET $N > 0$ BE GIVEN, THEN

$$\left| f(\varepsilon) g(\varepsilon) - \sum_{K=0}^{N} c_K \varepsilon^K \right| \longleftarrow \text{WTS} = o(\varepsilon^N)$$

$$= \left| f(\varepsilon) g(\varepsilon) - \sum_{K=0}^{N} \left(\sum_{i=0}^{K} a_i b_{K-i} \right) \varepsilon^K \right|$$

$$= \left| f(\varepsilon) g(\varepsilon) - \sum_{K=0}^{N} \sum_{i=0}^{K} \varepsilon^K a_i b_{K-i} \right| \qquad \underline{\text{TRICK}} \quad \varepsilon^K = \varepsilon^i \varepsilon^{K-i}$$

$$= \left| f(\varepsilon) g(\varepsilon) - \sum_{K=0}^{N} \sum_{i=0}^{K} (\varepsilon^i a_i)(\varepsilon^{K-i} b_{K-i}) \right|$$

BY $\underline{\text{HINT}}$ WITH $x_K = \varepsilon^K a_K$, $y_K = \varepsilon^K b_K$

$$= \left| f(\varepsilon) g(\varepsilon) - \left(\sum_{K=0}^{N} \varepsilon^K a_K \right) \left(\sum_{K=0}^{N} \varepsilon^K b_K \right) \right|$$

$$= \left| f(\varepsilon) g(\varepsilon) - f(\varepsilon) \sum_{K=0}^{N} \varepsilon^K b_K + f(\varepsilon) \sum_{K=0}^{N} \varepsilon^K b_K - \left(\sum_{K=0}^{N} \varepsilon^K a_K \right) \left(\sum_{K=0}^{N} \varepsilon^K b_K \right) \right|$$

$$\leq \underbrace{|f(\varepsilon)|}_{\substack{\leq C \\ \text{(BY ASSUMPTION)}}} \underbrace{\left| g(\varepsilon) - \sum_{K=0}^{N} \varepsilon^K b_K \right|}_{\substack{o(\varepsilon^N) \\ (\text{B/C } g \sim \sum \varepsilon^K b_K)}} + \underbrace{\left| f(\varepsilon) - \sum_{K=0}^{N} \varepsilon^K a_K \right|}_{\substack{o(\varepsilon^N) \\ (\text{B/C } f \sim \sum \varepsilon^K a_K)}} \underbrace{\left| \sum_{K=0}^{N} \varepsilon^K b_K \right|}_{\substack{\leq C \\ \text{(BY ASSUMPTION)}}}$$

$$\leq C \, o(\varepsilon^N)$$

$$= o(\varepsilon^N)$$

SEE NOTE ON NEXT PAGE:

NOTE TECHNICALLY THE HINT SHOULD HAVE SAID THAT

$$\left(\sum_{K=0}^{N} X_K\right)\left(\sum_{K=0}^{N} Y_K\right) = \sum_{M=0}^{N}\sum_{K=0}^{N} X_K Y_M ,$$

BUT NOTICE THAT ALL THE TERMS WITH $K+M > N$ CORRESPOND TO ε^{P} WITH $P > N$ AND HENCE ARE $o(\varepsilon^{N})$ ANYWAY,

SO IN THE END WE ONLY CARE ABOUT

$$\sum_{M=0}^{N} \sum_{\substack{K+M \leq N \\ (K \geq 0)}} X_K Y_M \quad \text{WHICH BECOMES} \quad \sum_{M=0}^{N} \sum_{K=0}^{M} X_K Y_{M-K}$$