SOLUTIONS

MATH 200 – MIDTERM 2

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Name: ____

Instructions: Welcome to Midterm 2! You have 50 minutes to take this exam, for a total of 100 points. **Do not open the exam until you're instructed to do so.** Remember that you are not only graded on the correctness of your answer, but also on the clarity and completeness of your proofs. Write in complete sentences whenever you can. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. Good luck, and may cardinality be with you! :)

Honor Code: I promise not to communicate or collaborate with anyone during the exam, and I will not use any books or notes or cheat sheets or personal electronic devices (including calculators).

Signature: ____

☐ Tonight is the deadline to switch to the Pass/Fail option. Please check this box if you're debating on whether to take this class Pass-Fail, and you would like me to look at your exam first and give you a rough estimate on how you're doing. Please be honest, as I cannot speed-grade 60 exams!

1		20
2		30
3		25
4		25
Total		100

Date: Friday, November 18, 2016.

1. *(20 points, 4 pts each;* [redacted]

Label the following statements as **TRUE (T)** or **FALSE (F)**. Any correct answer gives you 2 points, and any incorrect answer gives you 0 points. You do **NOT** get negative points for an incorrect answer, and you do **NOT** need to justify your answers.

F (a) $(A - B) \cup B = A$

F (b) $(\mathcal{P}(A))^c = \mathcal{P}(A^c)$ (Here $\mathcal{P}$ means Power Set)

T (c) If you toss a (fair) coin 4 times, the probability that at least one head occurs is $\frac{15}{16}$.

T (d) $A \cap B$ could be countable, even if A and B are uncountable.

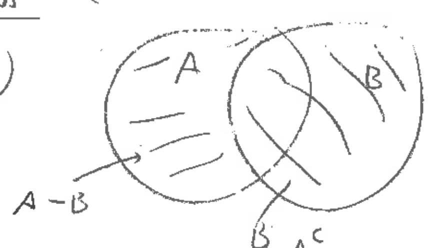
F (e) The general solution of the difference equation $u_{n+2} + 9u_n = 0$ is

$$u_n = A\,\mathrm{cosd}(3n) + B\,\mathrm{cosd}(3n).$$

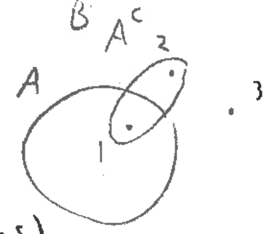
(e) FALSE IT'S $A\,3^N\,\mathrm{COSD}(N) + B\,3^N\,\mathrm{SIND}(N)$!

EXPLANATIONS (OPTIONAL)

(a) FALSE



$(A - B) \cup B = A \cup B$, NOT A



(b) FALSE

LET $A = \{1\}$, $A^C = \{2, 3\}$

THEN $\{1,2\}$ IS NOT A SUBSET OF A

SO $\{1,2\} \in (\mathcal{P}(A))^C$, BUT

$\{1,2\}$ IS NOT A SUBSET OF A^C, SO

$\{1,2\} \notin \mathcal{P}(A^C)$

HENCE $(\mathcal{P}(A))^C \not\subseteq \mathcal{P}(A^C)$

(c) TRUE BY THE COMPLEMENT RULE, THIS IS $1 - P(\text{ALL TAILS}) = 1 - \frac{1}{16} = \frac{15}{16}$

ONLY ONE CHOICE TTTT AMONG $2 \times 2 \times 2 \times 2 = 16$

(d) TRUE LET $A = (-\infty, 0]$, $B = [0, \infty)$ (UNCOUNTABLE). THEN $A \cap B = \{0\}$ COUNT

2. *(30 points,* [redacted]) Find all sequences u_n which satisfy the difference equation:

$$\begin{cases} u_{n+2} = 2u_{n+1} + 8u_n \\ u_0 = 1 \\ u_1 = -8 \end{cases}$$

WRITE AS $U_{N+2} - 2U_{N+1} - 8U_N = 0$

AUX $r^2 - 2r - 8 = 0$

$(r-4)(r+2) = 0$ (FACTOR OUT OR USE THE QUADRATIC EQ)

$r = -2, 4$

GEN SOL $U_N = A(-2)^N + B\,4^N$

NOW $U_0 = A + B = 1 \Rightarrow B = 1 - A$

SO $U_N = A(-2)^N + (1-A)\,4^N$

AND $U_1 = -2A + 4(1-A) = -2A + 4 - 4A = -6A + 4 = -8$

$\Rightarrow -6A = -12 \Rightarrow A = 2$

SO $A = 2$ AND $B = 1 - A = 1 - 2 = -1$

HENCE $\boxed{U_N = 2(-2)^N + (-1)\,4^N}$

3. [redacted] *(25 points)*

[redacted] Show that if $f : A \to B$ is one-to-one and S and T are subsets of A, then

$$f(S \cap T) = f(S) \cap f(T)$$

I'd like to remind you that $f(C) := \{f(x) \mid x \in C\}$.

[redacted]

($\subseteq$) SUPPOSE $y \in f(S \cap T)$

THEN $y = f(x)$, WHERE $x \in S \cap T$ (SO $x \in S$ AND $x \in T$)

SO ($y = f(x)$ WHERE $x \in S$) AND ($y = f(x)$ WHERE $x \in T$)

SO $y \in f(S)$ AND $y \in f(T)$

SO $y \in f(S) \cap f(T)$

($\supseteq$) SUPPOSE $y \in f(S) \cap f(T)$

THEN $y = f(s)$ WHERE $s \in S$ AND $y = f(t)$ WHERE $t \in T$

BUT THEN $f(s) = f(t)$, AND SO $s = t \in T$ <u>BECAUSE</u> f IS 1-1!

BUT THEN $s \in T$ (AND ALSO $s \in S$) AND SO $s \in S \cap T$

HENCE $y = f(\underbrace{s}_{\in S \cap T}) \in f(S \cap T)$

HENCE $y \in f(S \cap T)$

4. [redacted] *(25 points*[redacted]

(a) [redacted] Fix a function $f : \mathbb{R} \to \mathbb{R}$ and define a relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \Leftrightarrow f(x) = f(y)$$

Show that $\sim$ is an equivalence relation.

(b) *(10 points,* [redacted] Give an example of a function f such that $[0]$ (the equivalence class of 0) is uncountable. In your example, say what $[0]$ is and say why it's uncountable. You may use any facts covered in lecture/book/HW.

[redacted]!

(a) 1) $x \sim x$? IS $f(x) = f(x)$ FOR ALL x ? YES

2) $x \sim y \Rightarrow y \sim x$? SUPPOSE $f(x) = f(y)$, THEN $f(y) = f(y)$ ✓

3) $x \sim y$ AND $y \sim z \Rightarrow x \sim z$? IF $f(x) = f(y)$ AND $f(y) = f(z)$, THEN $f(x) = f(z)$ ✓

(B) LET $f(x) = 0$ (THE ZERO-FUNCTION) (ANY OTHER CONSTANT FUNCTION WORKS)

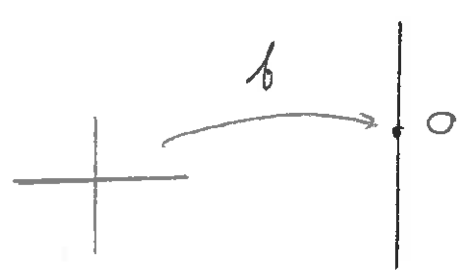
THEN $[0] = \{ y \in \mathbb{R} \mid y \sim 0 \}$

$= \{ y \in \mathbb{R} \mid f(y) = \underbrace{f(0)}_{0} \}$

$= \{ y \in \mathbb{R} \mid f(y) = 0 \}$

$= f^{-1}(\{0\})$

$= \mathbb{R}$ (SINCE EVERY y GETS MAPPED TO 0)



THEREFORE $[0] = \mathbb{R}$, BUT FROM LECTURE, $\mathbb{R}$ IS UNCOUNTABLE, AND HENCE