SOLUTIONS

MATH 200 – MIDTERM 1

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Instructions: Welcome to your midterm! You have 50 minutes to take this exam, for a total of 100 points. No books or notes or cheat sheets or personal electronic devices (including calculators) are allowed, and you may not communicate or collaborate with anyone during the exam. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. Good luck, and may modus ponens be with you! :)

1		[redacted]	10
2		[redacted]	25
3		[redacted]	20
4		[redacted]	30
5		[redacted]	15
Total		100	

Date: Wednesday, October 19, 2016.

1. *(10 points, 2 pts each)*

Label the following statements as **TRUE (T)** or **FALSE (F)**. Any correct answer gives you 2 points, and any incorrect answer gives you 0 points. You do **NOT** get points off for an incorrect answer, and you do **NOT** need to justify your answers. Please clearly mark your answer in the space to the left of each item.

T (a) If $P = P(x, y)$ is a proposition, then

$$[\exists y \, \forall x \, P(x, y)] \Rightarrow [\forall x \, \exists y \, P(x, y)]$$

F (b) If P and Q are propositions, then

$$\sim (P \wedge Q) \equiv ((\sim P) \wedge (\sim Q))$$

T (c) The following argument is valid

$$\begin{aligned} P &\Rightarrow Q \\ &\sim Q \\ \therefore &\sim P \end{aligned}$$

F (d) The negation of "All the students know discrete math" is "No students know discrete math".

F (e) The rule $f : \mathbb{Q} \to \mathbb{Z}$ given by $f\left(\frac{a}{b}\right) = ab$ is a function.

EXPLANATIONS (FOR YOU ONLY)

(a) T IN THE FIRST STATEMENT, THERE IS A y^* THAT MAKES $P(x, y^*)$ TRUE FOR ALL x.

IN THE SECOND STATEMENT, [redacted] GIVEN x, LET $y = y^*$ THEN $P(x, y^*)$ IS TRUE

(THE SECOND STATEMENT IS MORE GENERAL THAN THE FIRST)

(b) F IT SHOULD BE $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$

(c) T THAT IS JUST PROOF BY CONTRAPOSITION

(d) F THE NEGATION IS "THERE IS A STUDENT WHO DOESN'T KNOW DISCRETE MATH"

(e) F EVEN THOUGH $\frac{2}{3} = \frac{4}{6}$, WE HAVE $2 \times 3 \neq 4 \times 6$

2. *(25 points)*

(a) *(10 points)* Prove that that if n^2 is divisible by 5, then n is divisible by 5. You may use any facts discussed in lecture or the book or the homework.

(b) *(15 points)* Using (a), prove that $\sqrt{5}$ is irrational.

(a) SUPPOSE N^2 IS DIVISIBLE BY 5,
THEN $5 \mid N^2 = N \times N$, BUT SINCE 5 IS PRIME,
$5 \mid N$ OR $5 \mid N$, HENCE $5 \mid N$

AND SO N IS DIVISIBLE BY 5.

(HERE WE USED THE FACT THAT IF $p \mid ab$ & p IS PRIME,
THEN $p \mid a$ OR $p \mid b$)

(b) SUPPOSE $\sqrt{5}$ IS RATIONAL, THEN $\sqrt{5} = \frac{a}{b}$ FOR $a, b \in \mathbb{Z}$

WLOG, WE CAN ASSUME THAT a, b HAVE NO FACTORS IN COMMON

BUT $\sqrt{5} = \frac{a}{b} \Longleftrightarrow a = \sqrt{5}\, b \Longleftrightarrow a^2 = 5b^2 \quad (*)$

BUT SINCE $a^2 = 5b^2$, a^2 IS DIVISIBLE BY 5,

AND SO, BY (a), a IS DIVISIBLE BY 5

AND HENCE $a = 5k$ FOR SOME $k \in \mathbb{Z}$

BUT THEN, BY $(*)$, WE GET $(5k)^2 = 5b^2$

$$25k^2 = 5b^2$$
$$b^2 = 5k^2 = 5L \text{ FOR } L = k^2 \in \mathbb{Z}$$

AND SO b^2 IS DIVISIBLE BY 5,

AND SO, BY (a), b IS DIVISIBLE BY 5.

BUT THEN BOTH a & b ARE DIVISIBLE BY 5, AND SO THEY HAVE A FACTOR [illegible] $\Rightarrow\Leftarrow$ HENCE BY CONTRADICTION, $\sqrt{5}$ IS IRRATIONAL.

3. (20 POINTS) FIND THE REMAINDER WHEN

2017^{2017} IS DIVIDED BY 13.

HINT USE THE FACT THAT 2017 IS PRIME (I KNOW!!!) AS WELL AS THE FACT THAT

$$2017 = 166 \times 12 + 3 \quad \text{AND}$$

$$2017 = 155 \times 13 + 2$$

BY FERMAT'S LITTLE THEOREM, SINCE 13 IS PRIME,

$$2017^{13} \equiv 2017 \mod 13$$

BUT SINCE 2017 AND 13 ARE RELATIVELY PRIME (THEY ARE BOTH DISTINCT PRIMES), WE GET

$$2017^{12} \equiv 1 \mod 13$$

IN PARTICULAR, $2017^{2017} \equiv \underbrace{\left(2017^{12}\right)}_{\equiv 1}{}^{166} \times 2017^{3} = 2017^{3} \mod 13$

(WHERE WE USED $2017 = 166 \times 12 + 3$)

FINALLY, SINCE $2017 = 155 \times 13 + 2$, $2017 \equiv 2 \mod 13$

SO $2017^{3} \equiv 2^{3} \equiv 8 \mod 13$

SINCE $0 \leq 8 < 13$, WE INDEED HAVE $2017^{2017} \mod 13 = 8$

IN THAT (SHORTER) LINE, THE FIRST PERSON IS A PROF AND THE
~~...~~ LAST PERSON IS A PROFESSOR, SO BY INDUCTION HYP, THERE
IS A STUDENT DIRECTLY IN FRONT OF A PROFESSOR.
SINCE IN BOTH CASES P_{N+1} IS TRUE, P_{N+1} IS TRUE, SO BY INDUCTION P_N IS TRUE FOR ALL N.

4. *(30 points)* Use induction on n to show that if n people (where $n \geq 2$) stand in line (say at Paresky for lunch) and the person at the very front of the line is a student and the person at the very back of the line is a professor, then somewhere in the line there is a student *directly* in front of a professor.

Hint: For the inductive step, do this by cases, depending on whether the second person in line is a professor or a student.

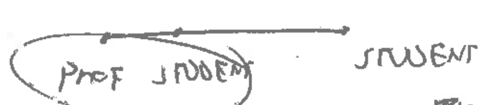
LET P_N BE THE PROPOSITION "IF N PEOPLE STAND IN LINE W/ STUDENT IN FRONT & PROF IN BACK, THEN THERE IS A STUDENT DIRECTLY IN FRONT OF PROF"

BASE CASE N=2 THEN BY ASSUMPTION, THERE ARE 2 PEOPLE, A STUDENT IN FRONT AND A PROF IN BACK

AND THAT STUDENT IS DIRECTLY IN FRONT OF PROF

INDUCTIVE STEP SUPPOSE P_N IS TRUE, SHOW P_{N+1} IS TRUE,
IE THERE ARE N+1 PEOPLE IN LINE, W/ A ~~...~~ STUDENT IN FRONT AND A PROF IN BACK.
NEED TO SHOW THAT THERE IS A STUDENT DIRECTLY IN FRONT OF A PROF.

CASE 1 PERSON 2 IS A STUDENT
THEN WE'RE DONE, B/C THE SECOND PERSON (THE STUDENT) IS DIRECTLY IN FRONT OF A PROF



CASE 2 PERSON 2 IS A PROF

THEN CONSIDER THE LINE FORMED BY PERSON 2, PERSON 3, ..., PERSON N+1

THE OTHER CASES ARE SIMILAR, AND SO, IN ALL 3 CASES, WE'VE SHOWN $\left\lfloor \frac{\lfloor \frac{x}{3} \rfloor}{3} \right\rfloor = \left\lfloor \frac{x}{9} \right\rfloor$, AND SO WE GET

$$\forall x \in \mathbb{R}, \quad \left\lfloor \frac{\lfloor \frac{x}{3} \rfloor}{3} \right\rfloor = \left\lfloor \frac{x}{9} \right\rfloor \quad \blacksquare$$

5. (15 *points*)

Warning: This problem is slightly harder, and I do not expect a lot of people to get it right!

Show that for all real numbers x,

$$\left\lfloor \frac{\lfloor \frac{x}{3} \rfloor}{3} \right\rfloor = \left\lfloor \frac{x}{9} \right\rfloor$$

Hint: Use the quotient remainder theorem and a proof by cases. In order to save you some time, you only need to show me how to do it for one of the cases.

LET $N = \left\lfloor \frac{x}{3} \right\rfloor$, THEN $N \leq \frac{x}{3} < N+1$

SO $3N \leq x < 3N+3$

NOW BY THE QUOTIENT-REMAINDER THEOREM WITH $d = 3$,

$N = 3q + r$ FOR $0 \leq r < 3$, THAT IS $r = 0, 1, 2$

q (NOT NINE)

CASE 1 $r = 0$, THEN $N = 3q$

SO $\left\lfloor \frac{\lfloor \frac{x}{3} \rfloor}{3} \right\rfloor = \left\lfloor \frac{3q}{3} \right\rfloor = \lfloor q \rfloor = q$ (SINCE $q \in \mathbb{Z}$) (*)

MOREOVER, FROM $3N \leq x < 3N+3$

WE GET $\frac{N}{3} \leq \frac{x}{9} < \frac{N}{3} + \frac{1}{3}$

NINE q

q $\frac{3q}{3} \leq \frac{x}{9} < \frac{3q}{3} + \frac{1}{3}$

q $q \leq \frac{x}{9} < q + \frac{1}{3} < q+1$

NINE

AND SO, BY DEFINITION, $\left\lfloor \frac{x}{9} \right\rfloor = q$ ← q

NINE

AND FINALLY WE GET $\left\lfloor \frac{x}{9} \right\rfloor = q = \left\lfloor \frac{\lfloor \frac{x}{3} \rfloor}{3} \right\rfloor$ ← BY (*) ✓