HW 9 - SOLUTIONS

PROBLEM 1

PART 1 OUTER SOLUTION (NEAR $X=1$)

STEP 1 ANSATZ $U^\varepsilon(x) = U^0(x) + \varepsilon U^1(x)$

PLUG INTO (ODE) $\varepsilon U^\varepsilon_{xx} + U^\varepsilon_x = 2x$

$$\varepsilon U^0_{xx} + \varepsilon^2 U^1_{xx} + U^0_x + \varepsilon U^1_x = 2x$$

$O(1)$ - TERMS $U^0_x = 2x \implies U^0(x) = x^2 + C \quad (C \in \mathbb{R})$

IMPOSE $U^0(1) = 1 \implies 1^2 + C = 1 \implies C = 0$

$\implies \boxed{U^0(x) = x^2}$ OUTER SOLUTION

PART 2 INNER SOLUTION (NEAR $X=0$)

STEP 2 CHANGE OF VARS $y = \dfrac{x}{\varepsilon^\alpha}$, $\overline{U^\varepsilon}(y) = U^\varepsilon(x)$

THEN $U^\varepsilon_x = \dfrac{dU^\varepsilon}{dx} = \dfrac{d\overline{U^\varepsilon}}{dy}\dfrac{dy}{dx} = \overline{U^\varepsilon_y}\left(\dfrac{1}{\varepsilon^\alpha}\right)$

AND $U^\varepsilon_{xx} = \dfrac{1}{\varepsilon^{2\alpha}} \overline{U^\varepsilon_{yy}}$ AND $2x = 2\varepsilon^\alpha y$

SO (ODE) BECOMES $\varepsilon\left(\dfrac{1}{\varepsilon^{2\alpha}}\right)\overline{U^\varepsilon_{yy}} + \dfrac{1}{\varepsilon^\alpha}\overline{U^\varepsilon_y} = 2\varepsilon^\alpha y$

$$\underset{(A)}{\varepsilon^{1-2\alpha}}\overline{U^\varepsilon_{yy}} + \underset{(B)}{\varepsilon^{-\alpha}}\overline{U^\varepsilon_y} - \underset{(C)}{2\varepsilon^\alpha y} = 0$$

STEP 3 <u>DOMINANT BALANCE</u>

<u>CASE 1</u> (A) ~ (B), (C) SMALLER $\Rightarrow$ $1-2\alpha = -\alpha$

$\Rightarrow$ $\alpha = 1$

THEN (A) ~ (B) $= \varepsilon^{-1}$, AND (C) $\sim \varepsilon^{1}$ SMALLER <u>BINGO</u>

<u>CASE 2</u> (A) ~ (C), (B) SMALLER $\Rightarrow$ $1-2\alpha = \alpha$

$\Rightarrow$ $\alpha = \frac{1}{3}$

BUT THEN (A) ~ (C) $= \varepsilon^{\frac{1}{3}}$ AND (B) $\sim \varepsilon^{-\frac{1}{3}}$, WHICH IS NOT SMALLER $\Rightarrow\Leftarrow$

<u>CASE 3</u> (B) ~ (C), (A) SMALLER $\Rightarrow$ $-\alpha = \alpha$ $\Rightarrow$ $\alpha = 0$

BUT THIS JUST GIVES US THE OUTER SOL $\Rightarrow\Leftarrow$

THEREFORE <u>$\alpha = 1$</u>, SO $Y = \frac{x}{\varepsilon}$

AND OUR (ODE) BECOMES

$$\varepsilon^{-1} \overline{U}^{\varepsilon}_{YY} + \varepsilon^{-1} \overline{U}^{\varepsilon}_{Y} - 2\varepsilon Y = 0$$

$$\Rightarrow \boxed{\overline{U}^{\varepsilon}_{YY} + \overline{U}^{\varepsilon}_{Y} - 2\varepsilon^2 Y = 0} \quad \text{(ODE)}$$

STEP 4 <u>ANSATZ</u> $\overline{U}^{\varepsilon}(Y) = \overline{U}^{0}(Y) + \varepsilon \overline{U}^{1}(Y) + \dots$

$$\overline{U}^{0}_{YY} + \varepsilon \overline{U}^{1}_{YY} + \overline{U}^{0}_{Y} + \varepsilon \overline{U}^{1}_{Y} - \varepsilon^2 Y = 0$$

<u>O(1) – TERMS</u> $\overline{U}^{0}_{YY} + \overline{U}^{0}_{Y} = 0$ <u>AUX</u> $r^2 + r = 0$

$r(r+1) = 0$

$\Rightarrow r = 0$ OR $r = -1$

$$\Rightarrow \overline{U}^{0}(Y) = Ae^{0Y} + Be^{(-1)Y}$$

$$\overline{U}^{0}(Y) = A + Be^{-Y}$$

<u>IMPOSE</u> $\overline{U}^{0}(0) = 1 \Rightarrow A + Be^{-0} = 1 \Rightarrow A + B = 1 \Rightarrow B = (1-A)$

so $\bar{U}^\circ(Y) = A + (1-A)e^{-Y}$

$= A + e^{-Y} - Ae^{-Y}$

$= A(1-e^{-Y}) + e^{-Y}$ INNER SOLUTION

$$\boxed{\bar{U}^\circ(Y) = A(1-e^{-Y}) + e^{-Y}}$$

PART 3) MATCHING

STEP 5 HERE WE'LL USE METHOD 1 (MATCHING IN ASYMPTOTIC LIMITS); WE'LL USE METHOD 2 IN THE NEXT PROBLEM ANYWAY

REQUIRE $\lim_{X \to 0} U^\circ(X) = \lim_{Y \to \infty} \bar{U}^\circ(Y)$

$$\lim_{X \to 0} X^2 = \lim_{Y \to \infty} A(1-e^{-Y}) + e^{-Y}$$

$$0 = A(1-0) + 0$$

$A = 0$

AND SO $\bar{U}^\circ(Y) = e^{-Y}$

STEP 6 THEREFORE $U^*(X) = U^\circ(X) + \bar{U}^\circ(Y) - \underbrace{\text{COMMON PART}}_{= \lim_{X \to 0} U^\circ(X) = 0}$

$= X^2 + e^{-Y}$

$$\boxed{U^*(X) = X^2 + e^{-\frac{X}{\varepsilon}}}$$

PROBLEM 2

PART 1 — OUTER SOLUTION (NEAR $x=1$)

STEP 1 — ANSATZ $\quad U^{\varepsilon}(x) = U^{0}(x) + \varepsilon U^{1}(x)$

PLUG INTO (ODE) $\quad \varepsilon U^{\varepsilon}_{xx} + U^{\varepsilon}_{x} + U^{\varepsilon} = 0$

$$\varepsilon U^{0}_{xx} + \varepsilon^{2} U^{1}_{xx} + U^{0}_{x} + \varepsilon U^{1}_{x} + U^{0} + \varepsilon U^{1} = 0$$

O(1) — TERMS $\quad U^{0}_{x} + U^{0} = 0 \Rightarrow U^{0}_{x} = -U^{0}$

$$U^{0}(x) = Ae^{-x}$$

IMPOSE $\quad U^{0}(1) = 1 \Rightarrow Ae^{-1} = 1 \Rightarrow A = e$

so $\quad U^{0}(x) = e\,e^{-x} = e^{1-x}$

$O(\varepsilon)$ — TERMS $\quad U^{0}_{xx} + U^{1}_{x} + U^{1} = 0$

$$\Rightarrow U^{1}_{x} + U^{1} = -U^{0}_{xx} = -e^{1-x} = -e\,e^{-x}$$

TO SOLVE THIS, USE UNDETERMINED COEFFICIENTS:

$$U^{1}(x) = y_{0}(x) + y_{p}(x)$$

$y_{0}(x)$ = SOLUTION TO $U^{1}_{x} + U^{1} = 0 \Rightarrow Ae^{-x}$

FOR $y_p(x)$, GUESS $\cancel{Ae^{-x}}$ ← COINCIDES W/ HOM SOL

GUESS $\quad A\underline{x}\,e^{-x}$

PLUG INTO $\quad U^{1}_{x} + U^{1} = -e\,e^{-x}$

$$(Axe^{-x})_x + (Axe^{-x}) = -e e^{-x}$$

$$A\cancel{e^{-x}} \cancel{- Axe^{-x}} \cancel{+ Axe^{-x}} = -e\cancel{e^{-x}}$$

$$\Rightarrow \underline{A = -e}$$

$$y_p(x) = -exe^{-x}$$

so $U^1(x) = y_0(x) + y_p(x) = Ae^{-x} - exe^{-x}$

IMPOSE $U^1(1) = 0 \Rightarrow Ae^{-1} - ee^{-1} = 0 \Rightarrow \frac{A}{e} = 1 \Rightarrow \underline{A = e}$

so $U^1(x) = ee^{-x} - exe^{-x} = e^{1-x}(1-x)$

CONCLUSION OUTER SOLUTION:
$$\begin{cases} U^0(x) = e^{1-x} \\ U^1(x) = (1-x)e^{1-x} \end{cases}$$

$\boxed{\text{PART 2}}$ INNER SOLUTION (NEAR $x=0$)

STEP 2 CHANGE OF VARS $y = \frac{x}{\varepsilon^\alpha}$, $\overline{U}^\varepsilon(y) = U^\varepsilon(x)$

THEN $U^\varepsilon_x = \frac{1}{\varepsilon^\alpha}\overline{U}^\varepsilon_y$, $U^\varepsilon_{xx} = \frac{1}{\varepsilon^{2\alpha}}\overline{U}^\varepsilon_{yy}$

so (ODE) BECOMES $\varepsilon\left(\frac{1}{\varepsilon^{2\alpha}}\right)\overline{U}^\varepsilon_{yy} + \frac{1}{\varepsilon^\alpha}\overline{U}^\varepsilon_y + \overline{U}^\varepsilon = 0$

$$\underbrace{\varepsilon^{1-2\alpha}\overline{U}^\varepsilon_{yy}}_{A} + \underbrace{\varepsilon^{-\alpha}\overline{U}^\varepsilon_y}_{B} + \underbrace{\overline{U}^\varepsilon}_{C} = 0$$

STEP 3 DOMINANT BALANCE

CASE 1 $A \sim B$, C small $\quad 1-2\alpha = -\alpha \Rightarrow \alpha = 1$

$A \sim B \sim \varepsilon^{-1}$, $C \sim 1$ small BINGO !

CASE 2 $\quad A \sim C$, B SMALL $\quad 1-2\alpha = 0 \Rightarrow \alpha = \frac{1}{2}$

THEN $\quad A \sim C \sim 1$, BUT THEN $B \sim \varepsilon^{-\frac{1}{2}}$ IS NOT SMALL

CASE 3 $\quad B \sim C$, A SMALL $\quad -\alpha = 0 \Rightarrow \alpha = 0$, BUT THIS JUST GIVES US OUTER!

THEREFORE $\underline{\alpha = 1}$ AND SO $Y = \frac{X}{\varepsilon}$ AND (ODE) BECOMES

$$\varepsilon^{-1}\overline{U}^{\varepsilon}_{YY} + \varepsilon^{-1}\overline{U}^{\varepsilon}_{Y} + \overline{U}^{\varepsilon} = 0$$

$$\boxed{\overline{U}^{\varepsilon}_{YY} + \overline{U}^{\varepsilon}_{Y} + \varepsilon\overline{U}^{\varepsilon} = 0} \quad (\text{ODE})'$$

STEP 4 $\quad$ ANSATZ $\quad \overline{U}^{\varepsilon}(Y) = \overline{U}^{0}(Y) + \varepsilon\overline{U}^{1}(Y) + \dots$

PLUG ANSATZ INTO (ODE)'

$$\overline{U}^{0}_{YY} + \varepsilon\overline{U}^{1}_{YY} + \overline{U}^{0}_{Y} + \varepsilon\overline{U}^{1}_{Y} + \varepsilon\overline{U}^{0} + \varepsilon^{2}\overline{U}^{1} = 0$$

O(1)-TERMS $\quad \overline{U}^{0}_{YY} + \overline{U}^{0}_{Y} = 0 \quad$ AUX $\quad r^2 + r = 0$

$r = 0$ OR -1

$$\overline{U}^{0}(Y) = A + Be^{-Y}$$

IMPOSE $\quad \overline{U}^{0}(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$

SO $\overline{U}^{0}(Y) = A - Ae^{-Y} = A(1-e^{-Y})$

TO FIGURE OUT A, USE MATCHING, AND HERE WE'LL USE METHOD 1 AGAIN

MATCHING IN ASYMPTOTIC LIMITS $\quad$ REQUIRE $\lim_{X\to 0} U^{0}(X) = \lim_{Y\to\infty} \overline{U}^{0}(Y)$

$$\lim_{X\to 0} e^{1-X} = \lim_{Y\to\infty} A(1-e^{-Y})$$

so $e = A(1-0) = A$

$\underline{A = e}$

AND $\boxed{\overline{U^0}(y) = e(1-e^{-y})} = e - e^{1-y}$

$\underline{O(\varepsilon)\text{-TERMS}}$ $\quad \overline{U^1_{yy}} + \overline{U^1_y} + \overline{U^0} = 0$

$$\Rightarrow \overline{U^1_{yy}} + \overline{U^1_y} = -\overline{U^0} = -e + e e^{-y}$$

TO SOLVE THIS, AGAIN USE UNDET COEFFICIENTS!

$$\overline{U^1}(y) = y_0(y) + y_p(y) = \underbrace{A_1 + Be^{-y}}_{\text{SOL TO } \overline{U^1_{yy}} + \overline{U^1_y} = 0} + y_p(y)$$

FOR $y_p(y)$, $\underline{\text{GUESS}}$ $\cancel{A + Be^{-y}}$ $\rightsquigarrow Ay + Bye^{-y}$

COINCIDES W/ HOM SOL

PLUG INTO $\quad (y_p)_{yy} + (y_p)_y = -e + e e^{-y}$

$$-Be^{-y} - \cancel{Be^{-y}} + \cancel{Bye^{-y}} + A + \cancel{Be^{-y}} - \cancel{Bye^{-y}} = -e + e e^{-y}$$

$$\underline{A} - \underline{Be^{-y}} = \underline{-e} + \underline{e e^{-y}}$$

$A = -e$ AND $-B = e$, SO $A = -e$ AND $B = -e$

THEREFORE $\quad y_p(y) = -ey - eye^{-y}$

AND $\overline{U^1}(y) = A + Be^{-y} - ey - y e^{1-y}$

IMPOSE $\overline{U^1}(0)=0 \Rightarrow A+B=0 \Rightarrow B=-A$

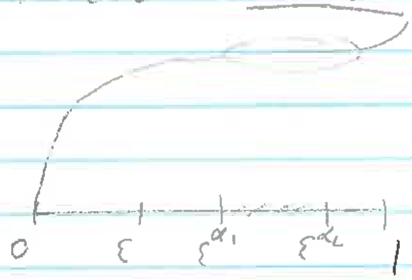
SO $\overline{U^1}(Y) = A - Ae^{-Y} - eY - Ye^{1-Y}$

$$\boxed{\overline{U^1}(Y) = A(1-e^{-Y}) - Ye(1+e^{-Y})}$$

PART 3 MATCHING

STEP 5 HERE WE CANNOT APPLY METHOD 2 SINCE $\lim_{Y\to\infty} \overline{U^1}(Y) = -\infty$,

SO WE HAVE TO RESORT TO METHOD 2 "MATCHING IN OVERLAPPING REGIONS"



SUPPOSE OVERLAPPING REGION IS $x\in(\varepsilon^{\alpha_1},\varepsilon^{\alpha_2})$ WITH $0<\alpha_2<\alpha_1<1$

LET $z = \dfrac{x}{\varepsilon^\beta}$ BE AN INTERMEDIATE VARIABLE (WHERE $0<\beta<1$)

THEN $U^0(x)+\varepsilon U^1(x) = e e^{-x} + \varepsilon(1-x)e^{1-x}$

$$= e e^{-\varepsilon^\beta z} + \varepsilon(1-\varepsilon^\beta z)e\, e^{-\varepsilon^\beta z}$$

$\left(Y = \dfrac{x}{\varepsilon} = \dfrac{\varepsilon^\beta z}{\varepsilon} = \varepsilon^{\beta-1} z\right)$

$\overline{U^0}(Y)+\varepsilon\overline{U^1}(Y) = e(1-e^{-Y}) + \varepsilon A(1-e^{-Y}) - \varepsilon Ye(1+e^{-Y})$

$$= e(1-e^{-\varepsilon^{\beta-1}z}) + \varepsilon A(1-e^{-\varepsilon^{\beta-1}z}) - \varepsilon^\beta z\, e(1+e^{-\varepsilon^{\beta-1}z})$$

HOWEVER, SINCE $\varepsilon^{\alpha_1}\le x\le\varepsilon^{\alpha_2}$, WE GET $\varepsilon^{\alpha_1}\le\varepsilon^\beta z\le\varepsilon^{\alpha_2}$,
SO BY SQUEEZE-1, WE HAVE $\varepsilon^\beta z\to 0$
AND SINCE $\varepsilon^{\alpha_1-1}\le\varepsilon^{\beta-1}z\le\varepsilon^{\alpha_2-1}$, BY SQUEEZE/LIMIT COMPARISON, WE GET $\varepsilon^{\beta-1}z\to\infty$

BUT NOW LETTING $\varepsilon^{\beta} z \to 0$ AND $\varepsilon^{\beta-1} z \to \infty$ IN

$$U^0(x) + \varepsilon U^1(x) = \overline{U^0}(y) + \varepsilon \overline{U^1}(y)$$, WE GET

$$e e^{-0} + \varepsilon(1-0) e e^{-0} = e(1-e^{-\infty}) + \varepsilon A(1-e^{-\infty}) - 0 e(1+e^{-\infty})$$

$$e + \varepsilon e = e + \varepsilon A$$

SO BY COMPARING THE $O(\varepsilon)$ - TERMS, WE GET $\underline{A = e}$

AND SO $\overline{U^1}(y) = e(1-e^{-y}) - y e(1+e^{-y})$

STEP 6 FINALLY: $U^*(x) = U^0(x) + \varepsilon U^1(x) + \overline{U^0}(y) + \varepsilon \overline{U^1}(y) - (\text{COMMON PARTS})$

$\hookrightarrow e + \varepsilon e$

$$U^*(x) = e^{1-x} + \varepsilon(1-x)e^{1-x} + e(\cancel{1} - e^{-\frac{x}{\varepsilon}}) + \varepsilon \cancel{e}(\cancel{1} - e^{-\frac{x}{\varepsilon}}) - x e(1+e^{-\frac{x}{\varepsilon}}) - \cancel{e} - \varepsilon \cancel{e}$$

$$\boxed{U^*(x) = e^{1-x} + \varepsilon(1-x)e^{1-x} - e^{1-\frac{x}{\varepsilon}} - \varepsilon e^{1-\frac{x}{\varepsilon}} - x e(1+e^{-\frac{x}{\varepsilon}})}$$