HW 8 - SOLUTIONS

PROBLEM 1

(a) LET V BE ARBITRARY

(TECHNICALLY V SUCH THAT $V = 0$ ON ∂W)

CONSIDER $g(\tau) = I[U + \tau V]$

$$= \int_W \frac{1}{2} |DU + \tau DV|^2 + f(U + \tau V) + |x|^2 \, dx$$

THEN $g'(\tau) = \int_W (DU + \tau DV) \cdot DV + f'(U + \tau V) V \, dx$

SINCE U IS A MINIMIZER, $g'(0) = 0$, AND SO

$$0 = g'(0) = \int_W DU \cdot DV + f'(U) V \, dx$$

$$\overset{IBP}{=} \int_W (-\operatorname{DIV}(DU)) V + f'(U) V \, dx$$

$$-\operatorname{DIV}(DU) = -\sum_{i=1}^{N} (U_{x_i})_{x_i} = -\sum_{i=1}^{N} U_{x_i x_i} = -\Delta U$$

$$= \int_W \left[-\Delta U + f'(U) \right] V \, dx$$

$$= 0$$

SINCE V WAS ARBITRARY, WE GET $\boxed{-\Delta U + f'(U) = 0}$

(b) LET θ BE ARBITRARY, AND CONSIDER

$$g(h) = \bar{I}[a^\circ, \theta^\circ + h\theta]$$

$$= \int_{\mathbb{R}^N} \frac{1}{2}|a^\circ|^2 |D\theta^\circ + h D\theta|^2 + \frac{1}{2} V(x)|a^\circ|^2 \, dx$$

$$g'(h) = \int_{\mathbb{R}^N} \cancel{\tfrac{1}{2}} |a^\circ|^2 \cancel{2}(D\theta^\circ + h D\theta) \cdot D\theta \, dx$$

SINCE θ° IS A MINIMIZER, $g'(0) = 0$, AND SO

$$0 = \int_{\mathbb{R}^N} (a^\circ)^2 \, D\theta^\circ \cdot D\theta \, dx$$

$$= -\int_{\mathbb{R}^N} \operatorname{DIV}\left((a^\circ)^2 D\theta^\circ\right) \theta \, dx$$

SINCE θ WAS ARBITRARY, WE GET

$$\boxed{-\operatorname{DIV}\left((a^\circ)^2 D\theta^\circ\right) = 0}$$

(c) LET $\boxed{L(p, z, x) = e^{-\varphi(x)} \left(\frac{1}{2}|p|^2 - f z\right)}$

THEN $D_p L(p, z, x) = e^{-\varphi(x)} p$

SO $D_p L(DU(x), U(x), x) = e^{-\varphi(x)} DU(x)$

SO $\operatorname{DIV}(D_p L) = \sum_{i=1}^N \left(e^{-\varphi(x)} U_{x_i}\right)_{x_i}$

$$= \sum_{i=1}^N e^{-\varphi(x)} (-\varphi_{x_i}(x)) U_{x_i} + e^{-\varphi(x)} U_{x_i x_i}$$

$$= e^{-\varphi(x)}\left(-D\varphi\cdot DU\right) + e^{-\varphi(x)}\Delta U$$

ALSO $\quad D_z L(p,z,x) = e^{-\varphi(x)}(-f)$,

SO $\quad D_z L(DU(x), U(x), x) = e^{-\varphi(x)}(-f(x))$

THEREFORE, THE (E-L) EQUATION ASSOCIATED TO L BECOMES:

$$-\mathrm{DIV}\left(D_p L(DU(x), U(x), x)\right) + L_z(DU(x), U(x), x) = 0$$

$$e^{-\varphi(x)} D\varphi\cdot DU - e^{-\varphi(x)}\Delta U - e^{-\varphi(x)} f(x) = 0$$

$$\cancel{e^{-\varphi}}\left(D\varphi\cdot DU - \Delta U - f\right) = 0$$

WOW! THIS DISAPPEARS!

$$\boxed{-\Delta U + D\varphi\cdot DU = f}$$

TA-DAAA!!!
(MAGICAL, ISN'T IT? ☺)

PROBLEM 2

ANSATZ $\quad U^\varepsilon(x) = U^0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon U^1\left(x, \frac{x}{\varepsilon}\right) + \cdots$

$$(U^K)' = U^K_x + \frac{1}{\varepsilon} U^K_y$$

$$(U^K)'' = U^K_{xx} + \frac{2}{\varepsilon} U^K_{xy} + \frac{1}{\varepsilon^2} U^K_{yy} \quad \text{(JUST AS USUAL)}$$

$$\varepsilon\left(U_0'' + \varepsilon U_1''\right) + \left(U_0' + \varepsilon U_1'\right) + U_0 + \varepsilon U_1 + \cdots = 0$$

$$\varepsilon U^0_{xx} + 2U^0_{xy} + \frac{1}{\varepsilon} U^0_{yy} + \varepsilon^2 U^1_{xx} + 2\varepsilon U^1_{xy} + U^1_{yy}$$

$$+ U^0_x + \frac{1}{\varepsilon} U^0_y + \varepsilon U^1_x + U^1_y + U_0 + \varepsilon U_1 = 0$$

(a) $O\left(\frac{1}{\varepsilon}\right)$-TERMS $\qquad U^0_{\tau\tau} + U^0_\tau = 0$

(THIS IS LIKE $y'' + y' = 0$

THE AUXILIARY EQUATION IS $r^2 + r = 0 \Rightarrow r(r+1) = 0$

$\Rightarrow r = 0$ OR $r = -1$)

$$\Rightarrow \quad U^0(x,\tau) = A(x)e^{-\tau} + B(x)e^{0\tau}$$

$$\boxed{U^0(x,\tau) = A(x)e^{-\tau} + B(x)}$$

(b) $O(1)$-TERMS

$$2U^0_{x\tau} + U^1_{\tau\tau} + U^0_x + U^1_\tau + U^0 = 0$$

ISOLATE U^1

$$U^1_{\tau\tau} + U^1_\tau = -2U^0_{x\tau} - U^0_x - U^0$$

$$= 2A'(x)e^{-\tau} - A'(x)e^{-\tau} - B'(x) - A(x)e^{-\tau} - B(x)$$

$$= (A'(x) - A(x))e^{-\tau} + (-B'(x) - B(x))\,1$$

HOWEVER, THE GENERAL SOLUTION TO $U^1_{\tau\tau} + U^1_\tau = 0$ IS $U^1(x,\tau) = A(x)e^{-\tau} + B(x)\,1$, SO THE TERMS $e^{-\tau}$ AND 1 CAUSE RESONANCE, THEREFORE WE WANT TO SELECT A AND B AS ABOVE TO KILL THE RESONANCE TERMS

SO SELECT A AND B SUCH THAT:

$$\begin{cases} A'(x) - A(x) = 0 \\ -B'(x) - B(x) = 0 \end{cases} \Rightarrow \begin{cases} A'(x) = A(x) \\ B'(x) = -B(x) \end{cases} \Rightarrow \begin{cases} A(x) = A_0 e^{x} \\ B(x) = B_0 e^{-x} \end{cases}$$

$(A_0, B_0 \in \mathbb{R})$

(c) THEREFORE, WE HAVE:

$$U^{\circ}(x,\tau) = A(x)e^{-\tau} + B(x)$$
$$= A_0 e^{x} e^{-\tau} + B_0 e^{-x}$$

SO $U^{\circ}(x) = U^{\circ}\left(x, \frac{x}{\varepsilon}\right) = A_0 e^{x} e^{-\frac{x}{\varepsilon}} + B_0 e^{-x}$

NOW USING $U^{\circ}(0) = 0$, WE GET $A_0(1)(1) + B_0(1) = 0$

$$\Rightarrow A_0 + B_0 = 0$$
$$\Rightarrow B_0 = -A_0$$

SO $U^{\circ}(x) = A_0 e^{x-\frac{x}{\varepsilon}} - A_0 e^{-x}$

AND FINALLY, USING $U^{\circ}(1) = 1$, WE HAVE

$$A_0 e^{1-\frac{1}{\varepsilon}} - A_0 e^{-1} = 1 \Rightarrow A_0 = \frac{1}{e^{1-\frac{1}{\varepsilon}} - e^{-1}}$$

AND SO:

$$\boxed{U^{\circ}(x) = \left(\frac{1}{e^{1-\frac{1}{\varepsilon}} - e^{-1}}\right) e^{x-\frac{x}{\varepsilon}} - \left(\frac{1}{e^{1-\frac{1}{\varepsilon}} - e^{-1}}\right) e^{-x}}$$

(UGLY, I KNOW !!)