

Math 379 – Homework 8

Peyam Tabrizian

Friday, November 11, 2016

Problem 1: For this problem, you may use the following integration-by-parts formula (assume the boundary terms are 0):

$$\int_W Df \cdot Dg \, d\mathbf{x} = - \int_W \operatorname{div}(Df) g \, d\mathbf{x}.$$

(a) Suppose u is a minimizer of

$$I[u] = \int_W \frac{1}{2} |Du|^2 + f(u) + |\mathbf{x}|^2 \, d\mathbf{x}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $W \subseteq \mathbb{R}^n$ and $f = f(s)$, $' = \frac{d}{ds}$.

Mimic the derivation of the Euler-Lagrange equation to show that u satisfy

$$-\Delta u + f'(u) = 0$$

(b) This part refers to Example 7 (An Eikonal/Continuity equation):

Suppose that a^0 and θ^0 minimize $I^0[a^0, \theta^0]$.

By doing a variation in θ^0 , show that θ^0 must satisfy the continuity equation

$$-\operatorname{div}((a_0)^2 D\theta^0) = 0$$

(c) Let f and ϕ be fixed functions. Find $L = L(p, z, x)$ so that the PDE

$$-\Delta u + D\phi \cdot Du = f$$

is the Euler-Lagrange equation corresponding to the functional $I[u] = \int_W L(Du, u, x) dx$.

Hint: First try to find a Lagrangian for the simpler equation $-\Delta u = f$. Then, to solve our original problem, multiply the Lagrangian you just found by a simple exponential term involving ϕ . Happy exploring :)

Problem 2: One last Chapter 3–type Expansion! :) Consider the following ODE on $(0, 1)$.

$$\begin{cases} \epsilon u''_\epsilon + u'_\epsilon + u_\epsilon = 0 \\ u_\epsilon(0) = 0, \quad u_\epsilon(1) = 1 \end{cases}$$

Apply the Ansatz

$$u^\epsilon(x) = u^0\left(x, \frac{x}{\epsilon}\right) + \epsilon u^1\left(x, \frac{x}{\epsilon}\right) + \dots$$

where $u^k = u^k(x, \tau)$.

- Find the $O\left(\frac{1}{\epsilon}\right)$ terms to get a formula for $u^0(x, \tau)$ in terms of constants $A(x)$ and $B(x)$.
- Find the $O(1)$ –terms to get an ODE of u^1 in terms of u^0 . Kill the resonance terms (what are they?) to find (the general form of) $A(x)$ and $B(x)$.
- Finally, impose the conditions $u^0(0) = 0$ and $u^0(1) = 1$ to find an explicit formula for $u^0(x) = u^0\left(x, \frac{x}{\epsilon}\right)$.