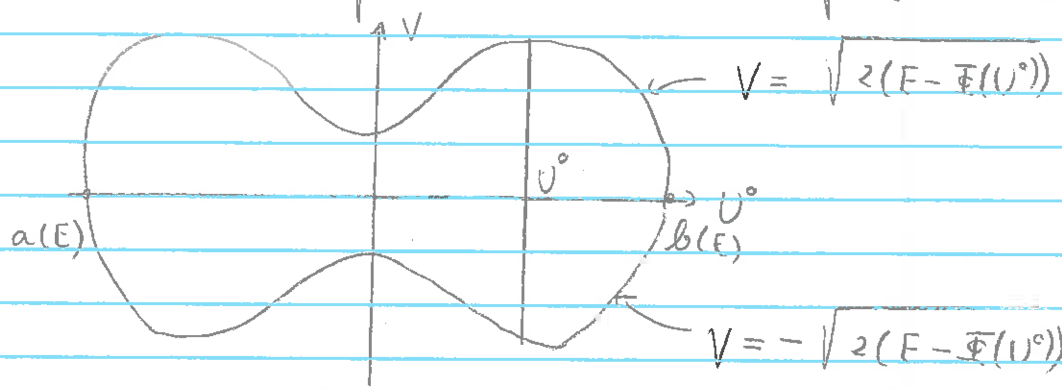
HW 7 - SOLUTIONS

PROBLEM 1

(a) NOTICE THAT $\frac{V^2}{2} + \Phi(U^\circ) \leq E$

$$\iff V^2 \leq 2(E - \Phi(U^\circ))$$

$$\iff -\sqrt{2(E-\Phi(U^\circ))} \leq V \leq \sqrt{2(E-\Phi(U^\circ))}$$



THEREFORE, FROM CALCULUS, THE AREA OF OUR REGION IS EQUAL TO

$$A = \int_{a(E)}^{b(E)} \Big(\underbrace{\sqrt{2(E-\Phi(U^\circ))}}_{\text{TOP HALF}} \Big) - \Big(\underbrace{-\sqrt{2(E-\Phi(U^\circ))}}_{\text{BOTTOM HALF}} \Big) \, dU^\circ$$

$$= 2\int_{a(E)}^{b(E)} \sqrt{2(E-\Phi(U^\circ))} \, dU^\circ$$

NOW FROM THE HINT, $\sqrt{2(E-\Phi(U^\circ))} = \omega^\circ U_m^\circ$, AND SO

$$A = 2\int_{a(E)}^{b(E)} \omega^\circ U_m^\circ \, dU^\circ$$

NOW LET'S DO A CHANGE OF VARIABLE $U^\circ = U^\circ(n)$, $n = (U^\circ)^{-1}(E)$

THEN $dU^\circ = \dfrac{dU^\circ}{dn}\, dn = U^\circ_n\, dn$

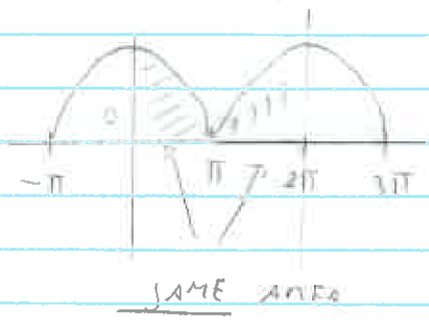
ALSO $b(E) = U^\circ(\pi)$ SO $\pi = (U^\circ)^{-1}(b(E))$

AND $a(E) = U^\circ(0)$, SO $0 = (U^\circ)^{-1}(a(E))$

AND SO WE GET $A = 2\displaystyle\int_0^\pi w^\circ\, U^\circ_n\, U^\circ_n\, dn$

$$= 2\int_0^\pi w^\circ (U^\circ_n)^2\, dn = 2\, w^\circ \int_0^\pi (U^\circ_n)^2\, dn$$

HOWEVER, BECAUSE $n \mapsto (U^\circ_n)^2$ IS EVEN AND 2π-PERIODIC



AREA FROM 0 TO 2π

$= 2 \times$ (AREA FROM 0 TO π)

WE GET $A = w^\circ\, 2\displaystyle\int_0^\pi (U^\circ_n)^2\, dn = w^\circ \int_0^{2\pi} (U^\circ_n)^2\, dn$

$$\boxed{A = \int_0^{2\pi} (w^\circ)(U^\circ_n)^2\, dn}$$

(b) FROM (a), WE HAVE

$$A = 2\int_{a(E)}^{b(E)} \sqrt{2(E-\Phi(U^\circ))}\, dU^\circ$$

NOW USING THE HINT, WE HAVE:

$$\frac{dA}{dE} = 2\int_{a(E)}^{b(E)} \frac{d}{dE}\left(\sqrt{2(E-\Phi(U^\circ))}\right) dU^\circ$$

$$+\ 2b'(E)\underbrace{\sqrt{2(E-\Phi(b(E))}}_{V(b(E))=0} - 2a'(E)\underbrace{\sqrt{2(E-\Phi(a(E))}}_{V(a(E))=0}$$

$$= 2\int_{a(E)}^{b(E)} \sqrt{2}\,\frac{1}{2\sqrt{E-\Phi(U^\circ)}}\, dU^\circ$$

$$= 2\int_{a(E)}^{b(E)} \frac{1}{\sqrt{2(E-\Phi(U^\circ))}}\, dU^\circ$$

$$= 2\int_{a(E)}^{b(E)} \frac{1}{\omega^\circ U^\circ_n}\, dU^\circ$$

CHANGE OF VARS:

$U^\circ = U^\circ(n)$

$dU^\circ = U^\circ_n\, dn$

$$= 2\int_0^{\pi} \frac{1}{\omega^\circ(E)\,\cancel{U^\circ_n}}\, \cancel{U^\circ_n}\, dn$$

$$= \frac{2}{\omega^\circ(E)}\int_0^{\pi} dn$$

$$= \frac{2\pi}{\omega^\circ(E)}$$

PROBLEM 2 $\qquad U_\varepsilon'' + w^2(\varepsilon t)\sin(U_\varepsilon) = 0 \quad (*)$

PRELIMINARY WORK $\qquad$ GUESS $\quad U^\varepsilon(t) = U\left(\frac{\theta(\varepsilon t,\varepsilon)}{\varepsilon}, \varepsilon t, \varepsilon\right),$

AND PLUG INTO $(*)$:

$$U_\varepsilon' = U_\eta\,\theta_\tau + U_\tau\,\varepsilon$$

$$U_\varepsilon'' = U_{\eta\eta}(\theta_\tau)^2 + U_{\eta\tau}\,\theta_\tau\,\varepsilon + U_\eta\,\theta_{\tau\tau}\,\varepsilon + U_{\tau\eta}\,\theta_\tau\,\varepsilon + U_{\tau\tau}\,\varepsilon^2$$

$$= U_{\eta\eta}(\theta_\tau)^2 + 2U_{\eta\tau}\,\theta_\tau\,\varepsilon + U_\eta\,\theta_{\tau\tau}\,\varepsilon + U_{\tau\tau}\,\varepsilon^2$$

SO $\qquad U_\varepsilon'' + w^2(\varepsilon t)\sin(U_\varepsilon) = 0$ BECOMES

$$U_{\eta\eta}(\theta_\tau)^2 + 2U_{\eta\tau}\,\theta_\tau\,\varepsilon + U_\eta\,\theta_{\tau\tau}\varepsilon + U_{\tau\tau}\varepsilon^2 + (w^0)^2\sin(U^\varepsilon) = 0$$

ANSATZ $\qquad U = U^0 + \varepsilon U^1 + \dots$

$\qquad\qquad \theta = \theta^0 + \varepsilon\,\theta^1 + \dots$

$$(U^0_{\eta\eta} + \varepsilon U^1_{\eta\eta})(\theta^0_\tau + \varepsilon\theta^1_\tau)^2 + 2(U^0_{\eta\tau} + \varepsilon U^1_{\eta\tau})(\theta^0_\tau + \varepsilon\theta^1_\tau)\varepsilon$$
$$+ (U^0_\eta + \varepsilon U^1_\eta)(\theta^0_{\tau\tau} + \varepsilon\theta^1_{\tau\tau})\varepsilon + (U^0_{\tau\tau} + \varepsilon U^1_{\tau\tau})\varepsilon^2 + w^2\,\underbrace{\sin(U^0 + \varepsilon U^1)}_{\text{TAYLOR}} = 0$$

$$U^0_{\eta\eta}(\theta^0_\tau)^2 + 2U^0_{\eta\eta}\,\theta^0_\tau\,\theta^1_\tau\,\varepsilon + \varepsilon U^1_{\eta\eta}(\theta^0_\tau)^2$$
$$+ 2U^0_{\eta\tau}\,\theta^0_\tau\,\varepsilon + U^0_\eta\,\theta^0_{\tau\tau}\,\varepsilon + w^2\sin(U^0) + \varepsilon U^1\cos(U^0)(w^0)^2$$

$O(1)$ $\qquad U^0_{\eta\eta}(w^0)^2 + w^2\sin(U^0) = 0 \qquad (w^0 = \theta^0_\tau)$

$O(\varepsilon)$ $\qquad 2U^0_{\eta\eta}\,w^0 w^1 + U^1_{\eta\eta}(w^0)^2 + 2U^0_{\eta\tau}\,w^0 + U^0_\eta\,w^0_\tau$
$$+ (w^0)^2 U^1\cos(U^0) = 0$$

$\qquad\qquad (w^1 = \theta^1_\tau)$

(a) $O(1)$ $U^0_{\eta\eta}\cancel{(w^0)^2} + \cancel{(w^0)^2}\sin(U^0) = 0$ $\leftarrow$ $w = \theta^0_T = w^0$

$$U^0_{\eta\eta} + \sin(U^0) = 0 \quad (*)$$

NOW LET $E(T,\eta) = \frac{1}{2}(U^0_\eta)^2 - \cos(U^0)$

THEN $\frac{dE}{d\eta} = U^0_\eta U^0_{\eta\eta} + \sin(U^0)U^0_\eta$

$$= U^0_\eta\underbrace{\left(U^0_{\eta\eta} + \sin(U^0)\right)}_{0, \text{ BY } (*)}$$

(b) DIFF $(*)$ $U^0_{\eta\eta} + \sin(U^0) = 0$

$$\downarrow \text{ DIFF}$$

$$U^0_{\eta\eta\eta} + \cos(U^0)U^0_\eta = 0$$

$$\downarrow W = U^0_\eta$$

$$W_{\eta\eta} + \cos(U^0)W = 0 \quad (LIN)$$

NOW $O(\varepsilon)$ SAYS (AGAIN, USING $w = \theta^0_T = w^0$)

$$2U^0_{\eta\eta}w^0w^1 + U^1_{\eta\eta}(w^0)^2 + 2U^0_{\eta T}w^0 + U^0_\eta w^0_T + (w^0)^2U^1\cos(U^0) = 0$$

ISOLATE U^1:

$$U^1_{\eta\eta}(w^0)^2 + (w^0)^2U^1\cos(U^0) = -2U^0_{\eta\eta}w^0w^1 - 2U^0_{\eta T}w^0 - U^0_\eta w^0_T$$

MULTIPLY BY $W = U^0_\eta$ (PERIODIC!) AND INTEGRATE W.R.T. η ON $[0, 2\pi)$

$$\int^{2\pi}(U^1)(w^0)^2W_{\eta\eta} + (w^0)^2U^1\cos(U^0)W\,d\eta \quad \leftarrow \text{INTEGRATED BY PARTS}$$

$$= \int_0^{2\pi} -2U^0_{\eta\eta}w^0w^1W - 2U^0_{\eta T}w^0W - U^0_\eta w^0_TW\,d\eta$$

$$\int_0^{2\pi} (W^0)^2 U^1 \Big(\overbrace{W_{\eta\eta} + \cos(U^0) W}^{0 \text{ (BY LIN)}} \Big) \, d\eta$$

$$= \int_0^{2\pi} -2 \underbrace{U^0_{\eta\eta} U^0_\eta}_{[(U^0_\eta)^2]_\eta} W^1 W^2 \; \underbrace{- 2 \underbrace{U^0_{\eta\tau} U^0_\eta}_{[(U^0_\eta)^2]_\tau W^0} W^0 - (U^0_\eta)^2 (W^0_\tau)}_{} \, d\eta$$

$$= W^1 W^2 \underbrace{\cancel{\left[(U^0_\eta)^2 \right]_0^{2\pi}}^{\,0}}_{\text{(PERIODICITY)}} \quad - \quad \frac{d}{d\tau} \left[(U^0_\eta)^2 W^0 \right]$$

SO $\quad 0 = -\dfrac{d}{d\tau} \displaystyle\int_0^{2\pi} (U^0_\eta)^2 W^0 \, d\eta$

$$\Longrightarrow \quad \frac{d}{d\tau} \int_0^{2\pi} (U^0_\eta)^2 W^0 \, d\eta = 0$$

$$\Longrightarrow \quad \boxed{\int_0^{2\pi} (U^0_\eta)^2 W^0 \, d\eta = C}$$

(c) $\quad \Theta^0_\tau = W$

AND SO $\quad \boxed{\Theta^0(\tau) = \int_0^{\tau} W(\tau') \, d\tau'}$