

Math 379 – Homework 7

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Problem 1: This example concerns Example 4: A nonlinear oscillator with damping.

- (a) Let A be the area of the region $\{(u_0, V) \in \mathbb{R}^2 \mid \frac{1}{2}(V^2) + \Phi(u_0) \leq E\}$. Show that

$$A = \int_0^{2\pi} \omega^0 (u_\eta^0)^2 d\eta$$

Hint: There is no Multivariable Calculus involved here, so don't go there! Just use single-variable calculus. Also use that (from lecture) $\sqrt{2(E - \Phi(u_0))} = \omega^0 u_\eta^0$, as well as the change of variable $u_0 = u_0(\eta)$. Finally, remember that ω^0 does not depend on η and that $(u_0)^2$ is even and 2π periodic.

- (b) Show that

$$\frac{dA}{dE} = \frac{2\pi}{\omega^0(E)}$$

Hint: Use the formula of A you found in (a) that involves square roots and $a(E)$ and $b(E)$, and differentiate this with respect to E . You may use the following differentiation formula (you need this because both the integrand and the limits depend on E):

$$\frac{d}{dE} \int_{a(E)}^{b(E)} f(x, E) dx = \int_{a(E)}^{b(E)} f_E(x, E) dx + b'(E) f(b(E), E) - a'(E) f(a(E), E)$$

Problem 2: This problem is similar to Examples 4 and 5 in lecture, so if you're stuck, just look back at what we did in those two examples. Consider

$$u_\epsilon'' + \omega^2(\epsilon t) \sin(u_\epsilon) = 0$$

where $u^\epsilon = u^\epsilon(t)$, $' = \frac{d}{dt}$, $\omega = \omega(s) > 0$.

Suppose u_ϵ has the form

$$u_\epsilon(t) = u\left(\frac{\theta(\epsilon t, \epsilon)}{\epsilon}, \epsilon t, \epsilon\right)$$

where $u = u(\eta, \tau, \epsilon)$, $\theta = \theta(\tau, \epsilon)$ and $\eta \mapsto u(\eta, \tau, \epsilon)$ is 2π -periodic

Apply the (usual) Ansatz

$$\begin{aligned} u &= u^0 + \epsilon u^1 + \dots \\ \theta &= \theta^0 + \epsilon \theta^1 + \dots \end{aligned}$$

Now just like the *WKB* method, **choose** θ_0 such that $\theta_\tau^0 = \omega$, and let $\omega_0 := \theta_\tau^0$, $\omega_1 := \theta_\tau^1$ (so in particular $\omega = \omega_0$)

(a) Let E be the energy

$$E(\tau, \eta) = \frac{1}{2} (u_\eta^0)^2 - \cos(u^0)$$

Show, using the $O(1)$ -terms, that $E = E(\tau)$.

(b) Differentiate the $O(1)$ -terms with respect to η and let $W = u_\eta^0$ to show that W solves a linear PDE (in W). Then multiply the $O(\epsilon)$ -terms by $W = u_\eta^0$ and integrate by parts and show that

$$\int_0^{2\pi} \omega^0 (u_\eta^0)^2 d\eta = C$$

(c) Finally, calculate θ^0 in terms of ω .

Note: Just like in Example 5 (Nonlinear Wave equation), it turns out that we can solve the three equations in (a), (b), (c) to find u_0 , ω_0 and θ_0