HW 6 - SOLUTIONS

PROBLEM 1

(a) FIRST OF ALL, $\forall K = 0, 1, 2, \ldots$

$$U_K' = \frac{d U_K(t, \varepsilon t)}{dt} = U_t^K + \varepsilon U_\tau^K$$

$$U_K'' = U_{tt}^K + \varepsilon U_{t\tau}^K + \varepsilon U_{\tau t}^K + \varepsilon^2 U_{\tau\tau}^K$$

$$= U_{tt}^K + 2\varepsilon U_{t\tau}^K + \varepsilon^2 U_{\tau\tau}^K$$

NOW $\quad U_\varepsilon'' + \varepsilon U_\varepsilon' + U_\varepsilon = 0$

$$\Rightarrow U_0'' + \varepsilon U_1'' + \varepsilon U_0' + \varepsilon^2 U_1' + U_0 + \varepsilon U_1 = 0$$

$$\Rightarrow U_{tt}^0 + 2\varepsilon U_{t\tau}^0 + \varepsilon^2 U_{\tau\tau}^0 + \varepsilon U_{tt}^1 + 2\varepsilon^2 U_{t\tau}^1 + \varepsilon^3 U_{\tau\tau}^1$$

$$+ \varepsilon U_t^0 + \varepsilon^2 U_\tau^0 + \varepsilon^2 U_t^1 + \varepsilon^3 U_\tau^1 + U^0 + \varepsilon U^1 = 0$$

O(1)-TERMS $\quad U_{tt}^0 + U^0 = 0 \quad$ (BUT $U^0 = U^0(t, \tau)$)

$$\Rightarrow U^0(t, \tau) = A(\tau)\cos(t) + B(\tau)\sin(t) \quad (*)$$

(b) O(ε)-TERMS $\quad 2U_{t\tau}^0 + U_{tt}^1 + U_t^0 + U^1 = 0$

$$U_{tt}^1 + U^1 = -2U_{t\tau}^0 - U_t^0 \quad \text{BY } (*)$$

$$= 2A'(\tau)\sin(t) - 2B'(\tau)\cos(t) + A(\tau)\sin(t) - B(\tau)\cos(t)$$

$$= (-2B'(\tau) - B(\tau))\cos(t) + (2A'(\tau) + A(\tau))\sin(t)$$

BUT NOTICE THAT THE GEN SOL TO $U^1_{tt} + U^1 = 0$
IS $A\cos(t) + B\sin(t)$,
SO IN THE PREVIOUS EQUATION, CHOOSE $A(T)$ AND $B(T)$ TO
KILL THE RESONANCE TERMS, THAT IS, SELECT A, B SO THAT

$$\begin{cases} 2A'(T) + A(T) = 0 \\ -2B'(T) - B(T) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} A'(T) = -\frac{1}{2}A(T) \\ B'(T) = -\frac{1}{2}B(T) \end{cases}$$

$$\Rightarrow \begin{cases} A(T) = A_0 e^{(-\frac{1}{2})T} \\ B(T) = B_0 e^{-(\frac{1}{2})T} \end{cases} \quad \text{FOR CONSTANTS } A_0 \,\&\, B_0$$

THEREFORE,

$$\begin{aligned} U_0(t) &= U_0(t, \varepsilon t) \\ &= A(\varepsilon t)\cos(t) + B(\varepsilon t)\sin(t) \\ &= A_0 e^{-\frac{\varepsilon t}{2}}\cos(t) + B_0 e^{-\frac{\varepsilon t}{2}}\sin(t) \end{aligned}$$

NOW SETTING $U_0(0) = 1$, WE GET $A_0(1)(1) + B_0(1)(0) = 1$

$$\underline{A_0 = 1}$$

MOREOVER $U_0'(t) = A_0\left(-\frac{\varepsilon}{2}\right)e^{-\frac{\varepsilon t}{2}}\cos(t) - A_0 e^{-\frac{\varepsilon t}{2}}\sin(t)$
$+ B_0\left(-\frac{\varepsilon}{2}\right)e^{-\frac{\varepsilon t}{2}}\sin(t) + B_0 e^{-\frac{\varepsilon t}{2}}\cos(t)$

so $U_0'(0) = -\frac{\varepsilon}{2} \overset{1}{\overset{\shortparallel}{A_0}} - A_0(0) + B_0(0) + B_0(1) = 1$

$\Rightarrow \quad -\frac{\varepsilon}{2} + B_0 = 1$

$\Rightarrow \quad B_0 = 1 + \frac{\varepsilon}{2}$

HENCE $\boxed{U_0(t) = e^{-\frac{\varepsilon t}{2}} \cos(t) + \left(1 + \frac{\varepsilon}{2}\right) e^{-\frac{\varepsilon t}{2}} \sin(t)}$

(c) THE AUXILIARY EQUATION OF $U_\varepsilon'' + \varepsilon U_\varepsilon' + U_\varepsilon = 0$
IS $r^2 + \varepsilon r + 1 = 0$

$\Rightarrow \quad r = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$

$= \frac{-\varepsilon}{2} \pm \frac{\sqrt{4-\varepsilon^2}}{2} i$

SO THE GENERAL SOLUTION TO THE ORIGINAL ODE IS :

$U_\varepsilon(t) = A e^{-\frac{\varepsilon}{2}t} \cos\left(\frac{\sqrt{4-\varepsilon^2}}{2} t\right) + B e^{-\frac{\varepsilon}{2}t} \sin\left(\frac{\sqrt{4-\varepsilon^2}}{2} t\right)$

NOW $U_\varepsilon(0) = 1 \Rightarrow \underline{A = 1}$

AND $U_\varepsilon'(t) = \left(-\frac{\varepsilon}{2}\right) A e^{-\frac{\varepsilon}{2}t} \cos\left(\frac{\sqrt{4-\varepsilon^2}}{2} t\right) - A e^{-\frac{\varepsilon}{2}t} \left(\frac{\sqrt{4-\varepsilon^2}}{2}\right) \sin\left(\frac{\sqrt{4-\varepsilon^2}}{2} t\right)$

$\left(-\frac{\varepsilon}{2}\right) B e^{-\frac{\varepsilon}{2}t} \sin\left(\frac{\sqrt{4-\varepsilon^2}}{2} t\right) + B e^{-\frac{\varepsilon}{2}t} \left(\frac{\sqrt{4-\varepsilon^2}}{2}\right) \cos\left(\frac{\sqrt{4-\varepsilon^2}}{2} t\right)$

so $U_\varepsilon'(0) = -\frac{\varepsilon}{2} \underset{\shortparallel_1}{A} + B\left(\frac{\sqrt{4-\varepsilon^2}}{2}\right) = 1$

$\Rightarrow \quad B\left(\frac{\sqrt{4-\varepsilon^2}}{2}\right) = 1 + \frac{\varepsilon}{2} = \frac{2+\varepsilon}{2}$

so $\quad B = \left(\frac{2+\varepsilon}{2}\right)\frac{2}{\sqrt{4-\varepsilon^2}} = \left(\frac{2+\varepsilon}{\sqrt{4-\varepsilon^2}}\right)$

AND SO:

$$U^{\varepsilon}(t) = e^{-\frac{\varepsilon}{2}t}\cos\left(\frac{\sqrt{4-\varepsilon^2}}{2}t\right) + \left(\frac{2+\varepsilon}{\sqrt{4-\varepsilon^2}}\right)e^{-\frac{\varepsilon}{2}t}\sin\left(\frac{\sqrt{4-\varepsilon^2}}{2}t\right)$$

NOTICE

$$\lim_{\varepsilon\to 0} U^{\varepsilon}(t) = e^{0}\cos\left(\frac{\sqrt{4}}{2}t\right) + \frac{2}{\sqrt{4}}e^{0}\sin\left(\frac{\sqrt{4}}{2}t\right)$$

$$= \cos(t) + \sin(t)$$

AND $\displaystyle\lim_{\varepsilon\to 0} U^{0}(t) = e^{0}\cos(t) + e^{0}\sin(t) = \cos(t) + \sin(t)$

so $\displaystyle\lim_{\varepsilon\to 0} U^{\varepsilon}(t) = \lim_{\varepsilon\to 0} U^{0}(t)$

PROBLEM 2

(a) SIMILAR TO PROBLEM 1(a), $\forall K = 0, 1, 2, \ldots$

$$\theta_K' = \theta_t^K + \frac{1}{\varepsilon}\theta_\tau^K$$

$$\theta_K'' = \theta_{tt}^K + \frac{2}{\varepsilon}\theta_{t\tau}^K + \frac{1}{\varepsilon^2}\theta_{\tau\tau}^K$$

NOW $\quad \theta_\varepsilon'' - \left[a + \frac{b}{\varepsilon}\cos\left(\frac{t}{\varepsilon}\right)\right]\sin(\theta_\varepsilon) = 0$

$$\Rightarrow \quad \theta_0'' + \varepsilon\,\theta_1'' - \left[a + \frac{b}{\varepsilon}\cos(\tau)\right]\sin(\theta_0 + \varepsilon\,\theta_1) = 0$$

$$\theta_{tt}^0 + \frac{2}{\varepsilon}\theta_{t\tau}^0 + \frac{1}{\varepsilon^2}\theta_{\tau\tau}^0 + \varepsilon\,\theta_{tt}^1 + 2\,\theta_{t\tau}^1 + \frac{1}{\varepsilon}\theta_{\tau\tau}^1$$

$$-\left[a + \frac{b}{\varepsilon}\cos(\tau)\right]\left(\sin(\theta_0) + \varepsilon\,\theta_1\cos(\theta_0)\right) = 0$$

HERE WE USED $f(a+h) = f(a) + h f'(a) + \cdots$
W/ $f = \sin(x)$,
$a = \theta_0$, $h = \varepsilon\,\theta_1$
(TAYLOR!)

$$\theta_{tt}^0 + \frac{2}{\varepsilon}\theta_{t\tau}^0 + \frac{1}{\varepsilon^2}\theta_{\tau\tau}^0 + \varepsilon\,\theta_{tt}^1 + 2\,\theta_{t\tau}^1 + \frac{1}{\varepsilon}\theta_{\tau\tau}^1$$

$$- a\sin(\theta_0) - a\,\varepsilon\,\theta_1\cos(\theta_0) - \frac{b}{\varepsilon}\cos(\tau)\sin(\theta_0) - b\,\theta_1\cos(\theta_0)\cos(\tau) = 0$$

$O\left(\frac{1}{\varepsilon^2}\right)$-TERMS $\quad \theta_{\tau\tau}^0 = 0$

MULTIPLY THIS BY θ^0 AND INTEGRATE OVER $[0, 2\pi]$: $\quad 0 = \int_0^{2\pi} (\theta_{\tau\tau}^0)(\theta^0)\, d\tau$

$$= \underbrace{\left[(\theta^0_\tau)\, \theta^0 \right]_0^{2\pi}}_{=\,0} - \int_0^{2\pi} (\theta^0_\tau)^2 \, d\tau$$

0 (BY PERIODICITY OF θ^0 AND θ^0_τ)

$$0 = -\int_0^{2\pi} \underbrace{(\theta^0_\tau)^2}_{\geq 0} \, d\tau$$

SO $\theta^0_\tau \equiv 0$, AND SO $\boxed{\theta^0 = \theta^0(t)}$

(b) $O\left(\frac{1}{\varepsilon}\right)$ - TERMS $\quad 2 \underbrace{\theta^0_{t\tau}}_{0} + \theta^1_{\tau\tau} - b\cos(\tau)\sin(\theta_0) = 0$

(SINCE θ^0 DOES NOT DEPEND ON τ)

$$\theta^1_{\tau\tau} = b\cos(\tau) \underbrace{\sin(\theta_0)}_{\text{DOES NOT DEPEND ON } \tau}$$

$$\Rightarrow \theta^1_\tau = b\sin(\tau)\sin(\theta_0) + A(t)$$

$$\Rightarrow \theta^1 = -b\cos(\tau)\sin(\theta_0) + \underbrace{A(t)\tau + B(t)}_{\text{SET } A = B = 0}$$

$$\Rightarrow \boxed{\theta^1(t,\tau) - b\cos(\tau)\sin(\theta_0(t))}$$

<u>$O(1)$ - TERMS</u> $\quad \theta^0_{tt} + 2\theta^1_{t\tau} - a\sin(\theta_0) - b\theta_1\cos(\theta_0)\cos(\tau) = 0$

INTEGRATE W.R.T. τ ON $[0, 2\pi]$

$$\int_0^{2\pi} \theta^0_{tt} + 2\theta^1_{t\tau} - a\sin(\theta_0) - b\,\theta_1\cos(\theta_0)\cos(\tau) = 0$$

$$\text{SO} \qquad 2\pi\, \theta^0_{tt} + \underbrace{2\theta^1_t(t,2\pi) - 2\theta^1_t(t,0)}_{=0 \text{ BY PERIODICITY}} - 2\pi a \sin(\theta_0) \qquad (*)$$
$$- b\cos(\theta_0) \int_0^{2\pi} \theta_1 \cos(\tau)\, d\tau = 0$$

(HERE WE USED THAT θ_0 DOES <u>NOT</u> DEPEND ON τ)

BUT SINCE $\theta^1 = -b\cos(\tau)\sin(\theta_0)$, WE GET

$$\int_0^{2\pi} \theta_1 \cos(\tau)\, d\tau = -b\sin(\theta_0) \underbrace{\int_0^{2\pi} \cos^2(\tau)}_{\pi}$$
$$= -b\pi \sin(\theta_0)$$

AND SO FROM $(*)$ WE GET

$$2\pi\, \theta^0_{tt} - 2\pi a \sin(\theta_0) + b^2 \pi \underbrace{\cos(\theta^0)\sin(\theta^0)}_{\frac{1}{2}\sin(2\theta^0)} = 0$$

$$\Rightarrow \left| \theta^0_{tt} - a\sin(\theta_0) + \frac{b^2}{4}\sin(2\theta^0) = 0 \right| \qquad \text{TA-DAAA!}$$