

Math 379 – Homework 6

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Problem 1: Apply the Ansatz

$$u_\epsilon(t) = u_0(t, \epsilon t) + \epsilon u_1(t, \epsilon t) + \dots$$

to the ODE

$$\begin{cases} u_\epsilon'' + \epsilon u_\epsilon' + u_\epsilon = 0 \\ u_\epsilon(0) = 1, u_\epsilon'(0) = 1 \end{cases} \quad (1)$$

Here $u_\epsilon = u_\epsilon(t)$ and $u_k = u_k(t, \tau)$ (here τ is a placeholder for ϵt)

- (a) Show that $u^0(t, \tau) = A(\tau) \cos(t) + B(\tau) \sin(t)$
- (b) Find an ODE for A and B and solve it, and use that to solve for $u_0(t) = u_0(t, \epsilon t)$. Impose the conditions $u_0(0) = 1$ and $u_0'(0) = 1$.

Hint: Select A and B to kill the resonance terms $\cos(t)$ and $\sin(t)$, just like we did for Duffing's equation.

- (c) Find the exact solution of the original ODE (1) and compare it with the solution u_0 you found in (b). Do we have

$$\lim_{\epsilon \rightarrow 0} u^\epsilon(t) = \lim_{\epsilon \rightarrow 0} u_0(t)?$$

Problem 2: [The Inverted Pendulum Problem] (from Physics). Consider the equation

$$\theta_\epsilon'' - \left[a + \frac{b}{\epsilon} \cos\left(\frac{t}{\epsilon}\right) \right] \sin(\theta_\epsilon) = 0,$$

where $a > 0$ and $b > 0$ are constants, and $\theta = \theta(t)$.

Apply the Ansatz:

$$\theta_\epsilon(t) = \theta^0\left(t, \frac{t}{\epsilon}\right) + \epsilon \theta^1\left(t, \frac{t}{\epsilon}\right) + \dots$$

where $\theta^k = \theta^k(t, \tau)$, and $\tau \mapsto \theta^k(t, \tau)$ is 2π periodic. Show that

(a) θ does not depend on τ , that is

$$\theta^0 = \theta^0(t)$$

Hint: For the \sin -term, use a (first-order) Taylor expansion at $\theta = \theta^0$ and $h = \epsilon\theta^1$. For the $\cos\left(\frac{t}{\epsilon}\right)$ -term, leave it as it is and write it as $\cos(\tau)$ (in particular, do **NOT** expand it out!) Now once you get the $O\left(\frac{1}{\epsilon^2}\right)$ -term, multiply your result by θ^0 and integrate by parts with respect to τ on $[0, 2\pi]$, similarly to what we did in Example 1 (Homogenization). Why are there no boundary terms in the integration?

(b)

$$\theta_{tt}^0 + \frac{b^2}{4} \sin(2\theta^0) - a \sin(\theta^0) = 0$$

Hint: Here find the $O\left(\frac{1}{\epsilon}\right)$ -terms and solve for θ^1 . Your solution may involve constants $A = A(t)$ and $B = B(t)$. Set them equal to 0. Then, find the $O(1)$ -terms and integrate whatever you found with respect to τ from 0 to 2π . Also remember your formula for θ^1 , and use that $\int_0^{2\pi} \cos^2(x) dx = \pi$ and that $\sin(2x) = 2 \sin(x) \cos(x)$.