HW 5 – SOLUTIONS

PROBLEM 1

STEP 1 START WITH

$$(7)\quad \begin{cases} -(aW')' = a' \\ W(0) = W(1) \end{cases} \quad \leftarrow \text{ HERE } ' = \frac{d}{dy}$$

INTEGRATE (7) WITH RESPECT TO y ON $[0, z]$ (WHERE $0 \leq z \leq 1$) TO GET

$$\int_0^z -(aW')'\, dy = \int_0^z a'\, dy$$

$$-a(z)\,W'(z) + a(0)\underbrace{W'(0)}_{=0} = a(z) - a(0)$$

(BY ASSUMPTION)

$$-aW' = a - a(0)$$

$$W'(z) = -1 + \frac{a(0)}{a(z)}$$

STEP 2 NOW INTEGRATE THIS WITH RESPECT TO z ON $[0, y]$ (WHERE $0 \leq y \leq 1$) TO GET:

$$\int_0^y W'(z)\, dz = \int_0^y -1 + \frac{a(0)}{a(z)}\, dz$$

$$W(y) - W(0) = -y + a(0)\int_0^y \frac{1}{a(z)}\, dz \qquad (*)$$

BUT PLUGGING IN $y=1$ IN $(*)$, WE GET:

$$\underbrace{W(1)-W(0)}_{0} = 1 - a(0)\int_0^1 \frac{1}{a(z)}\,dz$$

(BY ASSUMPTION)

$$\Rightarrow \quad a(0)\int_0^1 \frac{1}{a(z)}\,dz = 1$$

$$\Rightarrow \quad a(0) = \left(\int_0^1 \frac{dz}{a(z)}\right)^{-1}$$

THEREFORE $(*)$ GIVES US

$$\boxed{W(y) = W(0) - y + \frac{\int_0^y \frac{dz}{a(z)}}{\int_0^1 \frac{dz}{a(z)}}} \quad (**)$$

STEP 3 NOW BY DEFINITION OF $\bar{a}$, WE HAVE

$$\bar{a} = \int_0^1 a + 2W'a + Wa'\,dy \quad (***)$$

BUT INTEGRATING THE LAST TERM BY PARTS, WE GET:

$$\int_0^1 Wa'\,dy = \left[Wa\right]_0^1 - \int_0^1 W'a\,dy$$

$$= \underbrace{W(1)a(1) - W(0)a(0)}_{=0 \text{ (BY PERIODICITY)}} - \int_0^1 W'a\,dy = -\int_0^1 W'a\,dy$$

HENCE $(**)$ SIMPLIFIES TO READ

$$\bar{a} = \int_0^1 a + 2W'a - W'a \, dy$$

$$= \int_0^1 a + W'a \, dy = \int_0^1 a(1+W') \, dy$$

BUT BY THE DEFINITION $(**)$ OF W, WE HAVE

$$W' = -1 + \frac{\frac{1}{a(y)}}{\int_0^1 \frac{dz}{a(z)}}$$

SO $$a(1+W') = \cancel{a(y)} \left(\frac{\frac{1}{\cancel{a(y)}}}{\int_0^1 \frac{dz}{a(z)}} \right) = \left(\int_0^1 \frac{dz}{a(z)} \right)^{-1}$$

AND SO, FINALLY,

$$\bar{a} = \int_0^1 \underbrace{\left(\int_0^1 \frac{dz}{a(z)} \right)^{-1}}_{\text{CONSTANT}} dy = \left(\int_0^1 \frac{dz}{a(z)} \right)^{-1} \underbrace{\int_0^1 dy}_{1}$$

$$\boxed{\bar{a} = \left(\int_0^1 \frac{dz}{a(z)} \right)^{-1} = \left(\int_0^1 \frac{dy}{a(y)} \right)^{-1}}$$ (RELABEL z AS y)

BAZINGA

(TURN PAGE FOR (b))

(b) THE ODE (9) SAYS $\boxed{-\bar{a}\, U^{\circ}_{xx} = f(x)}$

INTEGRATE WRT z ON [0,y]

$$\Rightarrow U^{\circ}_{xx}(z) = -\frac{1}{\bar{a}} f(z)$$

$$\Rightarrow U^{\circ}_{x}(y) = -\frac{1}{\bar{a}} \int_0^y f(z)\,dz + A$$

INTEGRATE WRT y ON [0,x]

$$\Rightarrow \boxed{U^{\circ}(x) = -\left(\int_0^1 \frac{dz}{a(z)}\right) \int_0^x \left(\int_0^y f(z)\,dz\right) dy + Ax + B}$$

PROBLEM 2

(a) $$W(t) = A\cos(3t) + B\sin(3t)$$

$$W''(t) + W(t) = -\frac{1}{4}\cos(3t)$$

$$-9A\cos(3t) - 9B\sin(3t) + A\cos(3t) + B\sin(3t) = -\frac{1}{4}\cos(3t)$$

$$-8A\cos(3t) - 8B\sin(3t) = -\frac{1}{4}\cos(3t) + 0\sin(3t)$$

BY COMPARISON, $\begin{cases} -8A = -\frac{1}{4} \\ -8B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{32} \\ B = 0 \end{cases}$

$$\Rightarrow \boxed{W(t) = \frac{1}{32}\cos(3t)}$$

(b) HERE YOU HAVE TO GUESS

$$V(t) = A\underline{\underline{t}}\cos(t) + B\underline{\underline{t}}\sin(t)$$

BECAUSE THE $\cos(t)$ IN $-\frac{3}{4}\cos(t)$ COINCIDES WITH THE $\cos(t)$

IN $A\cos(t) + B\sin(t)$ (= THE SOLUTION OF $V'' + V = \underline{0}$)

(IF YOU TRIED $A\cos(t) + B\sin(t)$, YOU'D JUST GET $0 \ldots$)

NOW $$V''(t) + V(t) = -\frac{3}{4}\cos(t)$$

$$\left(A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t)\right)' + V(t) = -\frac{3}{4}\cos(t)$$

$$-A\sin(t) - A\sin(t) - \cancel{At\cos(t)} + B\cos(t) + B\cos(t) - \cancel{Bt\sin(t)} + \cancel{At\cos(t)} + \cancel{Bt\sin(t)} = -\frac{3}{4}\cos(t)$$

$$2B\cos(t) - 2A\sin(t) = -\frac{3}{4}\cos(t) + 0\sin(t)$$

SO $\begin{cases} 2B = -\frac{3}{4} \\ -2A = 0 \end{cases} \Rightarrow \begin{cases} B = -\frac{3}{8} \\ A = 0 \end{cases}$

$$\Rightarrow \boxed{V(t) = -\frac{3}{8}t\sin(t)}$$

(c) $$U_1(t) = y_0(t) + y_p(t)$$

GEN SOL TO $U_1'' + U_1 = 0$

PARTICULAR SOL TO $U_1'' + U_1 = -\cos^3(t) = -\frac{3}{4}\cos(t) - \frac{1}{4}\cos(3t)$

SO $y_0(t) = A\cos(t) + B\sin(t)$

AND $y_p(t) = V(t) + W(t) = -\frac{3}{8}t\sin(t) + \frac{1}{32}\cos(3t)$

SO $$U_1(t) = A\cos(t) + B\sin(t) - \frac{3}{8}t\sin(t) + \frac{1}{32}\cos(3t)$$

(d) $U_1(0) = A + \frac{1}{32} = 0 \Rightarrow \underline{A = -\frac{1}{32}}$

$$U_1'(t) = -A\sin(t) + B\cos(t) - \frac{3}{8}\sin(t) - \frac{3}{8}t\cos(t) - \frac{3}{32}\sin(3t)$$

$U_1'(0) = B = 0 \Rightarrow \underline{B = 0}$

HENCE

$$U_1(t) = -\frac{1}{32}\cos(t) - \frac{3}{8}t\sin(t) + \frac{1}{32}\cos(3t)$$
$$= -\frac{3}{8}t\sin(t) + \frac{1}{32}\left(\cos(3t) - \cos(t)\right)$$