HW 4 - SOLUTIONS

PROBLEM 1

STEP 1

FROM LECTURE, WE KNOW THAT

$$I[\varepsilon] \sim \sum_{k=0}^{\infty} (L_{2k}a)(0)\, \varepsilon^{\frac{k+1}{2}} = (L_0 a)(0)\sqrt{\varepsilon} + (L_2 a)(0)\,\varepsilon + (L_4 a)(0)\,\varepsilon^{3/2} + o(\varepsilon^{3/2})$$

WHERE $(L_0 a)(0) = a(0)\sqrt{\dfrac{2\pi}{|\varphi''(0)|}} = \sqrt{2\pi}$

FROM LECTURE $\qquad \varphi''(0) = -1$

$a(0) = 1$

NOW AGAIN FROM LECTURE, WITH $K=1$, WE HAVE:

$$(L_2 a)(0) = \frac{\tilde{a}'(0)}{1!} c_1 = \tilde{a}'(0) \int_{\mathbb{R}} \underbrace{y e^{-\frac{y^2}{2}}}_{\text{ODD}} dy = 0$$

AND AGAIN FROM LECTURE, WITH $K=2$, WE HAVE:

$$(L_4 a)(0) = \frac{\tilde{a}''(0)}{2} c_2 = \tilde{a}''(0)\frac{\sqrt{2\pi}}{2}$$

BY HINT

STEP 2

NOW BY DEFINITION, $\tilde{a}(y) = a(\Psi(y))\, m(\Psi(y))\, \Psi'(y)$
$= m(\Psi(y))\, \Psi'(y)$ (BECAUSE $a(x) = 1$)

SO $\tilde{a}'(y) = m'(\Psi(y))(\Psi'(y))^2 + m(\Psi(y))\Psi''(y)$

AND $\tilde{a}''(y) = m''(\Psi(y))(\Psi'(y))^3 + 2m'(\Psi(y))\Psi'(y)\Psi''(y) + m'(\Psi(y))\Psi'(y)\Psi''(y) + m(\Psi(y))\Psi'''(y)$

SO $\tilde{a}''(0) = m''(\Psi(0))(\Psi'(0))^3 + 2m'(\Psi(0))\Psi'(0)\Psi''(0) + m'(\Psi(0))\Psi'(0)\Psi''(0) + m(\Psi(0))\Psi'''(0)$

BUT REMEMBER THAT $\psi(0)=0$ (FROM LECTURE), AND SINCE $m \equiv 1$ NEAR 0, WE GET $m'(\psi(0)) = m'(0) = 0$, AND $m''(\psi(0)) = m''(0) = 0$

HENCE
$$\tilde{a}''(0) = 0\,(\psi'(0))^3 + 2\,(0)\,\psi'(0)\,\psi''(0) + (0)\,\psi'(0)\,\psi''(0) + \underbrace{m(0)}_{=1}\,\psi'''(0)$$
$$= \psi'''(0)$$

SO ALL WE NEED TO DO IS TO CALCULATE $\psi'''(0)$

STEP 3

USING OUR FUNDAMENTAL EQUATION

$$\varphi(\psi(y)) = -\frac{y^2}{2}$$

DIFF WRT y
$$\varphi'(\psi(y))\,\psi'(y) = -y$$

DIFF AGAIN
$$\varphi''(\psi(y))\,(\psi'(y))^2 + \varphi'(\psi(y))\,\psi''(y) = -1$$

DIFF AGAIN!
$$\varphi'''(\psi(y))\,(\psi'(y))^3 + 2\varphi''(\psi(y))\,\psi'(y)\,\psi''(y) + \varphi''(\psi(y))\,\psi'(y)\,\psi''(y) + \varphi'(\psi(y))\,\psi'''(y) = 0$$

PLUG $y=0$ AND USE $\psi(0)=0$ AND $\psi'(0) = \frac{1}{\sqrt{|\varphi''(0)|}} = 1$ (LECTURE)

WE GET:
$$\underbrace{\varphi'''(0)}_{0}\,(1)^2 + 2\,\underbrace{\varphi''(0)}_{=-1}\,\underbrace{\psi'(0)}_{1}\,\underbrace{\psi''(0)}_{0} + \underbrace{\varphi''(0)}_{=-1}\,\underbrace{\psi'(0)}_{1}\,\psi''(0) + \underbrace{\varphi'(0)}_{0}\,\psi'''(0) = 0$$

$$-\psi''(0) = 0 \Rightarrow \boxed{\psi''(0) = 0}$$

↳ BUT WE NEED $\psi'''(0)$!

DIFF AGAIN !!! (I KNOW !!)

$$\varphi''''(\psi(y))\,(\psi'(y))^4 + \varphi'''(\psi(y))\,3(\psi'(y))^2\psi''(y)$$

$$+\,2\varphi'''(\psi(y))(\psi'(y))^2\psi''(y) + 2\varphi''(\psi(y))\,(\psi''(y))^2 + 2\varphi''(\psi(y))\psi'(y) \times \psi'''(y)$$

$$+\,\varphi'''(\psi(y))(\psi'(y))^2\,\psi''(y) + \varphi''(\psi(y))\,(\psi''(y))^2$$

$$+\,\varphi''(\psi(y))\,\psi'(y)\,\psi'''(y) + \varphi''(\psi(y))\,\psi'(y)\,\psi'''(y)$$

$$+\,\varphi'(\psi(y))\,\psi''''(y) = 0$$

PLUG $y=0$ AND USE $\psi(0)=0,\ \psi'(0)=1,\ \psi''(0)=0$
$\varphi'(0)=0,\ \varphi''(0)=-1,\ \varphi'''(0)=0$

$$\varphi''''(0)\,(1)^4 + \overset{=0}{\varphi'''(0)}\,3(\psi'(0))^2\psi''(0)$$

$$+\,2\overset{=0}{\varphi'''(0)}(1)^2(0) + 2(\underbrace{\varphi''(0)}_{-1})(0)^2 + 2\underbrace{\varphi''(0)}_{=-1}\,\underbrace{\psi'(0)}_{1}\,\psi'''(0)$$

$$+\,\overset{=0}{\varphi'''(0)}\,(1)^2(0) + \varphi''(0)\,0^2$$

$$+\,\overset{=-1}{\varphi''(0)}\,1\,\psi'''(0) + \overset{=-1}{\varphi''(0)}\,1\,\psi'''(0) + \overset{=0}{\varphi'(0)}\,\psi''''(0) = 0$$

$$\varphi''''(0) - 2\psi'''(0) - \psi'''(0) - \psi'''(0) = 0$$

$$\psi'''(0) = \frac{1}{4}\varphi''''(0)$$

AND HENCE $\tilde{a}''(0) = \psi'''(0) = \frac{1}{4}\varphi''''(0)$

AND THUS $(L_4 a)(0) = \tilde{a}''(0)\,\frac{\sqrt{2\pi}}{2} = \frac{\sqrt{2\pi}\,\varphi''''(0)}{8}$

CONCLUSION $\boxed{I[\varepsilon] \sim \sqrt{2\pi}\sqrt{\varepsilon} + \frac{\sqrt{2\pi}\,\varphi''''(0)}{8}\,\varepsilon^{3/2} + o(\varepsilon^{3/2})}$

PROBLEM 2

(a) $$\frac{1}{Z_\varepsilon} = \int_{\mathbb{R}} e^{-\frac{H(x)}{\varepsilon}} dx$$

$$= 2\int_0^{\infty} e^{-\frac{H(x)}{\varepsilon}} dx \quad \text{(SINCE } H \text{ IS } \underline{\text{EVEN}}\text{)}$$

$$\sim 2(1)\, e^{-\frac{H(1)}{\varepsilon}} \sqrt{\frac{2\pi\varepsilon}{|H''(1)|}} + o(\sqrt{\varepsilon})$$ (BY LAPLACE'S METHOD WITH $a(x) \equiv 1$ AND $\varphi(x) = -H(x)$)

$$= \frac{2\sqrt{2\pi\varepsilon}}{\sqrt{H''(1)}} + o(\sqrt{\varepsilon}) \quad (H''(1) > 0)$$

(b) BY TAYLOR EXPANSION AROUND $X = 0$, WE HAVE

$$H(x) = H(0) + x\,\overset{\text{"}0}{H'(0)} + \frac{x^2}{2} H''(0) + \underbrace{O(|x|^2)}_{\text{IGNORE}}$$

$$H(x) = 1 - \frac{x^2}{2}|H''(0)|$$

AND SO $$\frac{\gamma^\varepsilon}{\sigma^\varepsilon} = \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}} \frac{1}{Z^\varepsilon} e^{\frac{H(x)}{\varepsilon}}$$

$$= \frac{1}{\varepsilon} e^{\cancel{-\frac{1}{\varepsilon}}} \frac{1}{Z^\varepsilon} e^{\cancel{\frac{1}{\varepsilon}} - \frac{x^2}{2\varepsilon}|H''(0)|}$$

$$= \frac{1}{\varepsilon Z^\varepsilon} e^{-\frac{x^2}{2\varepsilon}|H''(0)|}$$

(TURN PAGE)

THEREFORE,

$$\int_{-\delta}^{\delta} \frac{Z^{\varepsilon}}{\sigma^{\varepsilon}(x)}\,dx = \frac{1}{\varepsilon Z^{\varepsilon}} \int_{-\delta}^{\delta} e^{-\frac{x^2}{2\varepsilon}|H''(0)|}$$

$$\overset{\text{BY (a)}}{=} \frac{1}{\varepsilon}\left(\frac{2\sqrt{2\pi\varepsilon}}{\sqrt{H''(1)}}\right) \int_{-\delta}^{\delta} e^{-\frac{\left(\frac{x\sqrt{|H''(0)|}}{\sqrt{\varepsilon}}\right)^2}{2}}\,dx$$

$$y = \sqrt{\frac{|H''(0)|}{\varepsilon}}\,x$$

$$= \frac{1}{\varepsilon}\left(\frac{2\sqrt{2\pi\varepsilon}}{\sqrt{H''(1)}}\right) \frac{\sqrt{\varepsilon}}{\sqrt{|H''(0)|}} \int_{-\sqrt{\frac{|H''(0)|}{\varepsilon}}\,\delta}^{\sqrt{\frac{|H''(0)|}{\varepsilon}}\,\delta} e^{-\frac{y^2}{2}}\,dy$$

$$= \frac{2\sqrt{2\pi}}{\sqrt{H''(1)\,|H''(0)|}} \int_{-\sqrt{|H''(0)|}\frac{\delta}{\sqrt{\varepsilon}}}^{\sqrt{|H''(0)|}\frac{\delta}{\sqrt{\varepsilon}}} e^{-\frac{y^2}{2}}\,dy$$

$$\xrightarrow{\varepsilon \to 0} \frac{2\sqrt{2\pi}}{\sqrt{H''(1)\,|H''(0)|}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy}_{\sqrt{2\pi}}$$

$$= \frac{4\pi}{\sqrt{H''(1)\,|H''(0)|}}$$

$$= \frac{2}{\left(\frac{\sqrt{|H''(0)|\,H''(1)}}{2\pi}\right)}$$

$$= \boxed{2/K}$$