HW 3 - SOLUTIONS

PROBLEM 1

(a) $\quad (\varphi(x-ct))_t + 6(\varphi(x-ct))(\varphi(x-ct))_x + (\varphi(x-ct))_{xxx} = 0$

$$-c\varphi'(x-ct) + 6\varphi(x-ct)\varphi'(x-ct) + \varphi'''(x-ct) = 0$$

$$\boxed{-c\varphi' + 6\varphi\varphi' + \varphi''' = 0}$$

(b) $\quad -c\varphi' + 3(\varphi^2)' + \varphi''' = 0$

$$(-c\varphi + 3\varphi^2 + \varphi'')' = 0 \quad \text{ANTIDIFF.}$$

$$-c\varphi + 3\varphi^2 + \varphi'' = A$$

MULTIPLY BY φ'

$$-c\varphi\varphi' + 3(\varphi^2)\varphi' + (\varphi'')\varphi' = A\varphi'$$

$$\left(-c\frac{(\varphi^2)}{2} + (\varphi^3) + \frac{(\varphi')^2}{2}\right)' = (A\varphi)'$$

ANTIDIFF

$$-c\frac{(\varphi^2)}{2} + (\varphi^3) + \frac{(\varphi')^2}{2} = A\varphi + B$$

$$\Rightarrow \boxed{\frac{(\varphi')^2}{2} = -\varphi^3 + c\frac{(\varphi^2)}{2} + A\varphi + B}$$

(c) $\varphi' = -\varphi\sqrt{c-2\varphi}$

$$\frac{d\varphi}{ds} = -\varphi\sqrt{c-2\varphi}$$

$$\frac{d\varphi}{-\varphi\sqrt{c-2\varphi}} = ds$$

ANTIDIFF

$$\boxed{-\int \frac{d\varphi}{\varphi\sqrt{c-2\varphi}} = s + C} \quad (*)$$

(d) IF $\varphi = \frac{c}{2}\operatorname{sech}^2(\theta)$, THEN

$$\frac{d\varphi}{d\theta} = \frac{c}{2}\, 2\operatorname{sech}(\theta)\left(-\operatorname{sech}(\theta)\tanh(\theta)\right)$$

$$d\varphi = -c\operatorname{sech}^2(\theta)\tanh(\theta)\,d\theta$$

MOREOVER, $\sqrt{c-2\varphi} = \sqrt{c - c\operatorname{sech}^2(\theta)}$

$$= \sqrt{c}\sqrt{1-\operatorname{sech}^2(\theta)}$$

$$= \sqrt{c}\sqrt{\tanh^2(\theta)}$$

$$= \sqrt{c}\tanh(\theta)$$

SO $(*)$ BECOMES

$$-\int \frac{-c\operatorname{sech}^2(\theta)\tanh(\theta)\,d\theta}{\frac{c}{2}\operatorname{sech}^2(\theta)\sqrt{c}\tanh(\theta)} = s + C$$

$$\frac{2}{\sqrt{c}}\int d\theta = s + C \implies \frac{2}{\sqrt{c}}\theta = s + C \implies \boxed{s = \frac{2}{\sqrt{c}}\theta - C}$$

(e) $S = \frac{2}{\sqrt{c}} \theta - C$

$$\Rightarrow \frac{2}{\sqrt{c}} \theta = s + C$$

$$\Rightarrow \theta = \frac{\sqrt{c}}{2} (s + C)$$

HENCE $\varphi(s) = \frac{c}{2} \operatorname{SECH}^2(\theta)$

$$\boxed{\varphi(s) = \frac{c}{2} \operatorname{SECH}^2\left(\frac{\sqrt{c}}{2}(s + C)\right)}$$

(f) FINALLY,

$$u(x,t) = \varphi(x - ct)$$

$$\boxed{u(x,t) = \frac{c}{2} \operatorname{SECH}^2\left(\frac{\sqrt{c}}{2}(x - ct + C)\right)}$$

PROBLEM 2

(a) LET P_N BE "$\int_0^\infty e^{-t} t^{N-1} dt = (N-1)!$"

BASE CASE $N=1$ $\int_0^\infty e^{-t} = \left[-e^{-t}\right]_0^\infty = -0 + 1 = 1 = 0!$ ✓

INDUCTIVE STEP SUPPOSE P_N IS TRUE, $\int_0^\infty e^{-t} t^{N-1} dt = (N-1)!$

SHOW P_{N+1} IS TRUE, $\int_0^\infty e^{-t} t^N dt = N!$

BUT BY IBP, $\int_0^\infty e^{-t} t^N dt = \left[-e^{-t} t^N\right]_0^\infty - \int_0^\infty (-e^{-t}) N t^{N-1} dt$

$$= 0 + e^{0}(0) + N\int_0^\infty e^{-t} t^{N-1}\, dt$$

$$= N((N-1)!) \quad \text{(BY THE INDUCTIVE HYPOTHESIS)}$$

$$= N! \checkmark$$

(b) IF $s = \frac{t}{N}$, THEN $ds = \frac{dt}{N}$, SO $dt = N\,ds$,

SO $\int_0^\infty e^{-t} t^{N-1} dt$

$$= \int_0^\infty e^{-(Ns)} (Ns)^{N-1} N\, ds$$

$$= (N^{N-1}) N \int_0^\infty e^{-Ns} s^{N-1} ds$$

$$= N^N \int_0^\infty e^{-Ns} s^N \left(\frac{1}{s}\right) ds$$

$$= N^N \int_0^\infty e^{-Ns} e^{N\ln(s)} \left(\frac{1}{s}\right) ds$$

$$= N^N \int_0^\infty e^{-N(s-\ln(s))} \left(\frac{1}{s}\right) ds$$

(c) IF $\varepsilon = \frac{1}{N}$, THEN $N = \frac{1}{\varepsilon}$, SO THE ABOVE BECOMES

$$N^N \int_0^\infty e^{\frac{-(s-\ln(s))}{\varepsilon}} \left(\frac{1}{s}\right) ds = N^N \int_0^\infty e^{\frac{f(s)}{\varepsilon}} a(s)\, ds$$

WHERE $f(s) = -(s - \ln(s)) = \ln(s) - s$

$a(s) = \frac{1}{s}$

LET'S FIGURE OUT WHERE φ HAS A MAX:

$$\varphi'(s) = \frac{1}{s} - 1 \Rightarrow 0 \Rightarrow \frac{1}{s} = 1 \Rightarrow s = 1$$

MOREOVER $\varphi''(s) = -\frac{1}{s^2}$, SO $\varphi''(1) = -1 < 0$, SO φ ATTAINS

A MAX @ $s = 1$

HENCE, BY THE GENERAL LAPLACE METHOD WITH $x_0 = 1$

WE GET

$$(N-1)! = N^N \int_0^\infty e^{\frac{\varphi(s)}{\varepsilon}} a(s)\,ds = N^N e^{\frac{\varphi(1)}{\varepsilon}} \sqrt{\frac{2\pi\varepsilon}{|\varphi''(1)|}}\, a(1) + o(\sqrt{\varepsilon})$$

BY (a)

$\varepsilon = \frac{1}{N}$

$$= N^N e^{\frac{0-1}{\varepsilon}} \sqrt{\frac{2\pi\varepsilon}{|-1|}} \left(\frac{1}{1}\right) + o(\sqrt{\varepsilon})$$

$$= N^N e^{\frac{-1}{\left(\frac{1}{N}\right)}} \sqrt{2\pi\left(\frac{1}{N}\right)} + o\left(\sqrt{\frac{1}{N}}\right)$$

$$= N^N e^{-N} \sqrt{2\pi}\, N^{-\frac{1}{2}} + o\left(N^{-\frac{1}{2}}\right)$$

$$= \sqrt{2\pi}\, N^{N-\frac{1}{2}} e^{-N} + o\left(N^{-\frac{1}{2}}\right)$$

THEREFORE:

$$(N-1)! = \sqrt{2\pi}\, N^{N-\frac{1}{2}} e^{-N} + o\left(N^{-\frac{1}{2}}\right)$$

(d) MULTIPLYING THE ABOVE BY N TO GET:

$$N\left[(N-1)!\right] = \sqrt{2\pi}\, N\left(N^{N-\frac{1}{2}}\right) e^{-N} + N\, o\left(N^{-\frac{1}{2}}\right)$$

$$N! = \sqrt{2\pi}\, N^{N+\frac{1}{2}} e^{-N} + o\left(N^{\frac{1}{2}}\right)$$

THEREFORE,

$$\lim_{N\to\infty} \frac{\sqrt{2\pi}\, N^{N+\frac{1}{2}} e^{-N}}{N!}$$

$$= \lim_{N\to\infty} \frac{N! - o(N^{\frac{1}{2}})}{N!}$$

$$= \lim_{N\to\infty} 1 - \frac{o(\sqrt{N})}{N!}$$

$$= 1$$

SO $\sqrt{2\pi}\, N^{N+\frac{1}{2}} e^{-N} \sim N!$ AS $N \to \infty$

THAT IS, $N! \sim \sqrt{2\pi}\, N^{N+\frac{1}{2}} e^{-N}$ AS $N \to \infty$