Math 379 – Homework 3

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Friday, September 30, 2016

Problem 1: The purpose of this problem is to derive an explicit solution of a KdV equation. Consider the following equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

where $u = u(x,t)$, $x \in \mathbb{R}$, $t > 0$.

(a) Suppose u has the form $u(x,t) = \varphi(x - ct)$ for some $c \in \mathbb{R}$ and some function $\varphi = \varphi(s)$. Plug u into the KdV equation to obtain that φ must satisfy

$$-c\varphi' + 6\varphi\varphi' + \varphi''' = 0.$$

Here $' = \frac{d}{ds}$.

(b) First antidifferentiate the equation found in (a), then multiply the resulting equation by φ' and then antidifferentiate again to find that φ must satisfy

$$\frac{(\varphi')^2}{2} = -\varphi^3 + \frac{c}{2}\varphi^2 + A\varphi + B$$

for some constants A and B. For the rest of the problem, we'll set $A = B = 0$, so we get

$$\frac{(\varphi')^2}{2} = \varphi^2 \left(-\varphi + \frac{c}{2}\right).$$

from which we obtain $\varphi' = -\varphi\sqrt{c - 2\varphi}$ (here the minus-sign is for convenience)

(c) Using a separation of variables (like in Calculus II), show that we must have

$$s + C = -\int \frac{d\varphi}{\varphi\sqrt{c - 2\varphi}}$$

where C is an arbitrary constant.

(d) Use the Calculus II-type substitution $\varphi = \frac{c}{2}\operatorname{sech}^2(\theta)$ to show that

$$s = \frac{2}{\sqrt{c}}\theta - C$$

Note: You can assume that $\frac{d\operatorname{sech}(\theta)}{d\theta} = -\operatorname{sech}(\theta)\tanh(\theta)$ and that $1 - \operatorname{sech}^2(\theta) = \tanh^2(\theta)$, and that $\tanh(\theta) \geq 0$ (in this case).

(e) Use (d) and the definition of θ to conclude that

$$\varphi(s) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(s + C)\right)$$

(f) And finally conclude that

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct + C)\right).$$

Problem 2: The purpose of this problem is to prove Stirling's formula (which is very useful in probability; for example, it is used in the proof of the Central Limit Theorem):

$$n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}, \text{ as } n \to \infty,$$

where $f \sim g$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$, or, equivalently, $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 1$.

(a) Show using induction on $n = 1, 2, \cdots$ and integration by parts that (you can assume that the terms at $t = \infty$ are 0)

$$\int_0^\infty e^{-t}t^{n-1}dt = (n-1)!$$

(b) Use the substitution $s = \frac{t}{n}$ to show that the above integral is equal to

$$n^n \int_0^\infty e^{-n(s-\ln(s))} \frac{1}{s} ds.$$

(c) The general Laplace method says that if φ is a smooth function that has a max at x_0 with $\varphi'(x_0) = 0$ and $\varphi''(x_0) < 0$, and a is any smooth function (not necessarily with compact support), then, as $\epsilon \to 0$,

$$\int_0^\infty a(x) e^{\frac{\varphi(x)}{\epsilon}} dx = \sqrt{\frac{2\pi\epsilon}{|\varphi''(x_0)|}}\, e^{\frac{\varphi(x_0)}{\epsilon}}\, a(x_0) + o(\sqrt{\epsilon}).$$

Apply the general Laplace method to the integral in (b), with $\epsilon = \frac{1}{n}$, as well as the result in (a) to show that, as $n \to \infty$

$$(n-1)! = \sqrt{2\pi} n^{n-\frac{1}{2}} e^{-n} + o(n^{-\frac{1}{2}})$$

(d) Multiply the above identity by n on both sides and conclude, using the definition of $\sim$ that,

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}.$$

You may assume that $\sqrt{n} = o(n!)$ as $n \to \infty$