HW 2 - SOLUTIONS

PROBLEM 1 BY THE CHAIN RULE (WHERE $\varphi(x,y,t) = \tilde{\varphi}(\xi, y, \tau)$)

1) $$\frac{\partial \varphi}{\partial x} = \frac{\partial \tilde{\varphi}}{\partial \xi}\frac{\partial \xi}{\partial x} = \tilde{\varphi}_\xi \sqrt{\varepsilon} = (\sqrt{\varepsilon}\,\tilde{\varphi})_\xi = \tilde{\Psi}_\xi$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\left(\tilde{\varphi}_\xi \sqrt{\varepsilon}\right) = \left(\tilde{\varphi}_{\xi\xi}\right)\left(\frac{\partial \xi}{\partial x}\right)(\sqrt{\varepsilon}) = \tilde{\varphi}_{\xi\xi}\,\varepsilon = \sqrt{\varepsilon}\,\tilde{\Psi}_{\xi\xi}$$

$$\frac{\partial \varphi}{\partial y} = \tilde{\varphi}_y = \frac{1}{\sqrt{\varepsilon}}(\sqrt{\varepsilon}\,\tilde{\varphi})_y = \frac{1}{\sqrt{\varepsilon}}\tilde{\Psi}_y$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \tilde{\varphi}_{yy} = \frac{1}{\sqrt{\varepsilon}}(\sqrt{\varepsilon}\,\tilde{\varphi})_{yy} = \frac{1}{\sqrt{\varepsilon}}\tilde{\Psi}_{yy}$$

THEREFORE $$\Delta \varphi = \varphi_{xx} + \varphi_{yy} = 0$$

BECOMES $$\sqrt{\varepsilon}\,\tilde{\Psi}_{\xi\xi} + \frac{1}{\sqrt{\varepsilon}}\tilde{\Psi}_{yy} = 0$$

$\times\sqrt{\varepsilon}$

$$\boxed{\varepsilon\,\tilde{\Psi}_{\xi\xi} + \tilde{\Psi}_{yy} = 0}$$

2) SINCE $\dfrac{\partial \varphi}{\partial y} = \dfrac{1}{\sqrt{\varepsilon}}\tilde{\Psi}_y$, $\quad \varphi_y = 0 \Leftrightarrow \dfrac{1}{\sqrt{\varepsilon}}\tilde{\Psi}_y = 0 \Leftrightarrow \tilde{\Psi}_y = 0$

3) $$\frac{\partial \varphi}{\partial t} = \frac{\partial \tilde{\varphi}}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \tilde{\varphi}}{\partial \tau}\frac{\partial \tau}{\partial t}$$
$$= \tilde{\varphi}_\xi\left(-c_0\sqrt{\varepsilon}\right) + \tilde{\varphi}_\tau\,\varepsilon^{3/2}$$
$$= -c_0\,\tilde{\Psi}_\xi + \varepsilon\,\tilde{\Psi}_\tau$$

AND AS BEFORE $\frac{\partial \varphi}{\partial x} = \tilde{\Psi}_\xi$, $\frac{\partial \varphi}{\partial y} = \frac{1}{\sqrt{\varepsilon}} \tilde{\Psi}_Y$

THEREFORE $\varphi_t + \frac{1}{2}|\nabla \varphi|^2 = -(h - h_0)$ BECOMES

$$-c_0 \tilde{\Psi}_\xi + \varepsilon \tilde{\Psi}_\tau + \frac{1}{2}\left(|\tilde{\Psi}_\xi|^2 + \frac{1}{\varepsilon}|\tilde{\Psi}_Y|^2\right) = -(\tilde{h} - h^0)$$

$\times \varepsilon$

$$\boxed{-c_0 \varepsilon \tilde{\Psi}_\xi + \varepsilon^2 \tilde{\Psi}_\tau + \frac{1}{2}\left(\varepsilon|\tilde{\Psi}_\xi|^2 + |\tilde{\Psi}_Y|^2\right) = -\varepsilon(\tilde{h} - h^0)}$$

4) $\frac{\partial h}{\partial x} = \frac{\partial \tilde{h}}{\partial \xi}\frac{\partial \xi}{\partial x} = \tilde{h}_\xi \sqrt{\varepsilon}$

$\frac{\partial h}{\partial t} = \frac{\partial \tilde{h}}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \tilde{h}}{\partial \tau}\frac{\partial \tau}{\partial t} = \tilde{h}_\xi(-c_0\sqrt{\varepsilon}) + \tilde{h}_\tau \varepsilon^{3/2}$

$\frac{\partial \varphi}{\partial x} = \tilde{\Psi}_\xi$ (SEE 1))

$\frac{\partial \varphi}{\partial y} = \frac{1}{\sqrt{\varepsilon}} \tilde{\Psi}_Y$ (SEE 1))

HENCE $h_t + \varphi_x h_x = \varphi_y$ BECOMES

$$-c_0\sqrt{\varepsilon}\,\tilde{h}_\xi + \tilde{h}_\tau \varepsilon^{3/2} + \tilde{\Psi}_\xi \tilde{h}_\xi \sqrt{\varepsilon} = \frac{1}{\sqrt{\varepsilon}} \tilde{\Psi}_Y$$

$\times \sqrt{\varepsilon}$

$$\boxed{\tilde{h}_\tau \varepsilon^2 + \varepsilon(\tilde{\Psi}_\xi - c_0)\tilde{h}_\xi = \tilde{\Psi}_Y}$$

PROBLEM 2

$$\begin{cases} U_t^\varepsilon + \frac{1}{\varepsilon} U_x^\varepsilon = \dfrac{(V^\varepsilon)^2 - (U^\varepsilon)^2}{\varepsilon^2} \\ V_t^\varepsilon - \frac{1}{\varepsilon} V_x^\varepsilon = \dfrac{(U^\varepsilon)^2 - (V^\varepsilon)^2}{\varepsilon^2} \end{cases} \quad (*)$$

(a) ANSATZ
$$\begin{cases} U^\varepsilon = U^0 + \varepsilon U^1 + \cdots \\ V^\varepsilon = V^0 + \varepsilon V^1 + \cdots \end{cases}$$

PLUG ANSATZ INTO $(*)$:

$$U_t^0 + \varepsilon U_t^1 + \cdots + \frac{1}{\varepsilon}\left(U_x^0 + \varepsilon U_x^1 + \cdots\right)$$
$$= \frac{1}{\varepsilon^2}\left((V^0 + \varepsilon V^1 + \cdots)^2 - (U^0 + \varepsilon U^1 + \cdots)^2\right)$$

MULTIPLY BY ε^2 ON BOTH SIDES:

$$\varepsilon^2 U_t^0 + \varepsilon^3 U_t^1 + \cdots + \varepsilon U_x^0 + \varepsilon^2 U_x^1 + \cdots$$
$$= (V^0)^2 + \varepsilon 2 V^0 V^1 + \cdots - (U^0)^2 - \varepsilon 2 U^0 U^1 - \cdots$$

$$(U^0)^2 - (V^0)^2 + \varepsilon\left(U_x^0 - 2V^0V^1 + 2U^0U^1\right) + \varepsilon^2\left(U_t^0 + U_x^1 + \cdots\right) = 0$$

$O(1)$ - TERMS $\quad (U^0)^2 - (V^0)^2 = 0 \implies U^0 = V^0 \quad (\text{B/C } U^0, V^0 > 0)$

$O(\varepsilon)$ - TERMS $\quad U_x^0 - 2V^0V^1 + 2U^0U^1 = 0$

$$U_x^0 = 2(V^0V^1 - U^0U^1)$$
$$= 2(U^0V^1 - U^0U^1) \qquad (U^0 = V^0)$$
$$= 2U^0(V^1 - U^1)$$

(b) $$\left(U^\varepsilon + V^\varepsilon\right)_t + \frac{1}{\varepsilon}\left(U^\varepsilon - V^\varepsilon\right)_x = 0$$

$$\left(U^0 + \varepsilon U^1 + \dots + V^0 + \varepsilon V^1\right)_t + \frac{1}{\varepsilon}\left(U^0 + \varepsilon U^1 - V^0 - \varepsilon V^1 + \dots\right)_x = 0$$

$$(U^0)_t + \varepsilon (U^1)_t + \dots + (V^0)_t + \varepsilon (V^1)_t + \frac{1}{\varepsilon}\left((U^0)_x + \varepsilon (U^1)_x - (V^0)_x - \varepsilon (V^1)_x + \dots\right) = 0$$

MULTIPLY BY ε

$$\varepsilon\left((U^0)_t + (V^0)_t\right) + \varepsilon^2\left((U^1)_t + (V^1)_t\right)$$

$$+ (U^0)_x - (V^0)_x + \varepsilon\left((U^1)_x - (V^1)_x\right) + \dots = 0$$

O(1) - TERMS $\qquad (U^0)_x - (V^0)_x = 0$ (ALREADY KNOW THIS)

O(ε) - TERMS $\qquad (U^0)_t + (V^0)_t + (U^1)_x - (V^1)_x = 0$

$U^0 = V^0$

$$2(U^0)_t = (V^1 - U^1)_x$$

$$(U^0)_t = \frac{(V^1 - U^1)_x}{2} \qquad (*)$$

(c) BY (*),

$$(U^0)_t = \frac{\left((V^1) - (U^1)\right)_x}{2}$$

BUT BY $(U^0)_x = 2(U^0)(V^1 - U^1)$, WE GET

$$V^1 - U^1 = \frac{1}{2}\frac{(U^0)_x}{U^0}$$

AND SO $$(U^0)_t = \frac{1}{2}(V^1 - U^1)_x = \frac{1}{2}\left(\frac{1}{2}\frac{(U^0)_x}{U^0}\right)_x = \frac{1}{4}\left(\frac{(U^0)_x}{U^0}\right)_x = \frac{1}{4}\left[\ln(U^0)\right]_{xx}$$