

Math 379 – Homework 2

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Problem 1: Show that, in Example 3 (Derivation of the KdV equation), after the change of variable

$$\begin{cases} \xi := \sqrt{\epsilon}(x - c_0 t) \\ \tau := \epsilon^{\frac{3}{2}} t \\ \psi := \sqrt{\epsilon} \varphi \end{cases}$$

where $c_0 := \sqrt{h^0}$, the PDE for $\tilde{\psi}$ and \tilde{h} become

$$\begin{cases} \epsilon \tilde{\psi}_{\xi\xi} + \tilde{\psi}_{yy} = 0 & \text{at } y = 0 \\ \tilde{\psi}_y = 0 & \text{at } y = 0 \\ \epsilon^2 \tilde{\psi}_\tau - \epsilon c_0 \tilde{\psi}_\xi + \frac{1}{2} (\epsilon (\tilde{\psi}_\xi)^2 + (\tilde{\psi}_y)^2) = -\epsilon (\tilde{h} - h^0) & \text{at } y = \tilde{h} \\ \epsilon^2 \tilde{h}_\tau + \epsilon (\tilde{\psi}_\xi - c_0) \tilde{h}_\xi = \tilde{\psi}_y & \text{at } y = \tilde{h} \end{cases}$$

Here $\tilde{\psi}$ and \tilde{h} are ψ and h in the new variables, i.e. $\psi(x, y, t) = \tilde{\psi}(\xi, y, \tau)$ and $h(x, t) = \tilde{h}(\xi, \tau)$. It may also be useful to use $\tilde{\varphi}$, i.e. $\varphi(x, y, t) = \tilde{\varphi}(\xi, y, \tau)$.

Note: For sake of clarity, the original PDE are

$$\begin{cases} \Delta \varphi = 0 & \text{at } y = 0 \\ \varphi_y = 0 & \text{at } y = 0 \\ \varphi_t + \frac{1}{2} |\nabla \varphi|^2 = -(h - h^0) & \text{at } y = h \\ h_t + \varphi_x h_x = \varphi_y & \text{at } y = h \end{cases}$$

Problem 2: Let u^ϵ and v^ϵ solve the system

$$\begin{cases} u_t^\epsilon + \frac{1}{\epsilon} u_x^\epsilon = \frac{(v^\epsilon)^2 - (u^\epsilon)^2}{\epsilon^2} \\ v_t^\epsilon - \frac{1}{\epsilon} v_x^\epsilon = \frac{(u^\epsilon)^2 - (v^\epsilon)^2}{\epsilon^2} \end{cases} \quad (1)$$

Let our Ansatz be:

$$\begin{aligned} u^\epsilon &= u^0 + \epsilon u^1 + o(\epsilon) \\ v^\epsilon &= v^0 + \epsilon v^1 + o(\epsilon) \end{aligned}$$

And moreover suppose that $u^0 > 0$ and $v^0 > 0$

(a) Plug the Ansatz into (1) and show that

$$u^0 = v^0 \quad (2)$$

and

$$u_x^0 = 2u^0(v^1 - u^1). \quad (3)$$

(b) Notice that if we add the two equations of (1), we get that

$$(u^\epsilon + v^\epsilon)_t + \frac{1}{\epsilon}(u^\epsilon - v^\epsilon)_x = 0. \quad (4)$$

Plug the Ansatz into (4) and show that

$$u_t^0 = \frac{1}{2}(v^1 - u^1)_x$$

(c) Using the results of (a) and (b), show that u^0 is a solution of the nonlinear heat equation

$$w_t - \frac{1}{4}(\ln w)_{xx} = 0.$$

Hint: Use (3) to write $(v^1 - u^1)_x$ only in terms of u^0 and its derivatives. Also, it may be helpful to write out explicitly what $(\ln w)_{xx}$ is.